Math H54 Midterm 1 Nov 14, 2022

Name: ______________________________

I_n denote the identity matrix of size n. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with complex entries. The standard Hermitian form on $\mathbb{C}^n$ is given by

$$\langle z, w\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_n w_n$$

where $z = (z_1, \cdots, z_n)^t, w = (w_1, \cdots, w_n)^t$ are two column vectors. (superscript t stands for 'transpose')

1. (30 pts, 10 points each) True or False. If you think the answer is true, please justify your answer; if you think the answer is false, give a counter-example.

   (a) For any complex $n \times n$ matrix A, there exists an invertible $n \times n$ matrix C, such that $C^{-1}AC$ is a diagonal matrix.

   False. Ex: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

   (b) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation, and $\mathbb{C}^n$ be equipped with the standard Hermitian form. Assume there exists a basis of $\mathbb{C}^n$ consist of eigenvectors of T, then these basis vectors are orthogonal to each other.

   False. $T = 0$ $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   not orthogonal to each other.

spectral thm

$T: \mathbb{C}^n \to \mathbb{C}^n$ lin op.

there exist ONB of $\mathbb{C}^n$, and ONB are eigenvectors of T

$\Leftrightarrow$ T is normal.

   (c) If T and S are linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, and $TS = ST$. Then if $Tv = \lambda v$ for some $v \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, we also have $T(Sv) = \lambda Sv$.

   $T(Sv) = STv = S(\lambda v) = \lambda(Sv)$

   True.

2. (35 points) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear map, where

e_1, e_2, e_3 are std basis in $\mathbb{C}^3$.

$$Te_1 = e_1 + e_2$$
$$Te_2 = e_2 + e_3$$
$$Te_3 = e_3 + e_1$$

(a) (10 points): If $z = (2, 3, 1)^t$, what is $Tz =$?.

$$z = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 2 \cdot e_1 + 3 \cdot e_2 + e_3 .$$

$$T(2e_1 + 3e_2 + e_3) = 2\, T(e_1) + 3 \cdot T(e_2) + T(e_3)$$
$$= 2(e_1 + e_2) + 3(e_2 + e_3) + e_3 + e_1$$
$$= 3e_1 + 5e_2 + 4e_3 .$$

(b) (10 points): If $z = (z_1, z_2, z_3)^t$, what is $Tz =$?

$$T(z_1e_1 + z_2e_2 + z_3e_3) = z_1 T(e_1) + z_2 T(e_2) + z_3 T(e_3)$$
$$= z_1(e_1 + e_2) + z_2(e_2 + e_3) + z_3(e_3 + e_1)$$
$$= (z_1 + z_3)e_1 + (z_1 + z_2)e_2 + (z_2 + z_3)e_3.$$

Can you find invertible C s.t.

$$T \cdot C = C \cdot \underset{\substack{\uparrow \\ \text{diag}}}{\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}}$$

$$C = \begin{pmatrix} \underset{\tilde{e}_1}{(\;)} & \underset{\tilde{e}_2}{(\;)} & \underset{\tilde{e}_3}{(\;)} \end{pmatrix}$$

(c) (15 points): Can T be diagonalized? Namely, can you find a basis $\widetilde{e_i}$ of $\mathbb{C}^3$, such that $T\widetilde{e_i} = \lambda_i \widetilde{e_i}$ for some $\lambda_i \in \mathbb{C}$? Please give your reasoning. (Hint: you don't have to find $\widetilde{e_i}$ and λ_i explicitly.)

Two methods :

#1 recommended.

matrix of T is
$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = id + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(\lambda - T) = (\lambda - 1)^3 + 1$$

$\Rightarrow$ roots are all distinct

$$\lambda - 1 = \sqrt[3]{-1} = \begin{cases} -1 \\ -e^{2\pi i/3} \\ -e^{4\pi i/3} \end{cases}$$

$\Rightarrow$ T is diagonalizable.

#2 : check if T is normal, if normal, then diagonalizable.

$$T^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \text{check } T^*T = T \cdot T^*.$$

$$T^*T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

similarly do $TT^* = \cdots$

$$\bullet \quad z = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

$$T(e_1, e_2, e_3) = (e_1 + e_2,\ e_2 + e_3,\ e_3 + e_1)$$

$$= (e_1, e_2, e_3)\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$Tz = T(e_1, e_2, e_3)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = (e_1, e_2, e_3)\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}}$$

$$= (e_1\, e_2\, e_3)\begin{pmatrix} z_1 + z_3 \\ z_1 + z_2 \\ z_2 + z_3 \end{pmatrix} = (z_1 + z_3)e_1 + (z_1 + z_2)e_2 + (z_2 + z_3)e_3$$

$$R_3 \cdot R_2 \cdot R_1 T \cdot = \overset{\text{diagonal.}}{D}$$

3. (35 points) Let $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a symmetric bilinear form, for any $z, w \in \mathbb{C}^n$, we have

$$B(z,w) = z^t[B]w = \sum_{i=1}^{n}\sum_{j=1}^{n} z_i B_{ij} w_j,$$

where $[B]$ is a symmetric matrix with entries B_{ij}.

(a) (10 points) if $\det([B]) \neq 0$, can you always find an invertible matrix C, such that $C^t[B]C = I_n$? Please explain.

Yes. By the inertia thm for $\mathbb{C}$- bilinear form.

- any symm $\mathbb{C}$- matrix B, $\exists$ invertible C, s.t.

$$C^t \cdot B \cdot C = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{pmatrix} \}r$$

- Here, $r = n$ because

$$\det(C^t \cdot B \cdot C) = \det(C^t) \cdot \det(B) \cdot \det C = \det(B) \cdot [\det(C)]^2 \neq 0$$

(b) (10 points) if $\det([B]) \neq 0$, is it true that for any $v \neq 0$, we have $B(v,v) \neq 0$? Please explain.

False : $$(1 \; i)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 0$$

(c) (15 points) Let the matrix $[B]$ be given by

$$[B] = \begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}$$

Find a matrix C such that $C^t[B]C$ is diagonal. Please show your steps.

e.g.

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{C^t}\begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{C} = \begin{pmatrix} 4+2i & 0 \\ 0 & 4-2i \end{pmatrix}$$

Remark : what is a square matrix ?

what can you learn by finding the eigen value?

- A matrix M is $\begin{pmatrix} - & - & - & - \\ - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$. , a bunch of numbers.

it doesn't mean anything yet.

- A linear transformation

$$T : V \to V$$

is a 'meaningful' obj, and one can ask for its eigenvalue & eigenvectors.

- We can represent a linear transformation using a matrix, by choosing a basis of V

say $e_1, \cdots, e_n$ is a basis.

then any vector. $v = (e_1, \cdots, e_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad x_i \in \mathbb{C}$

suppose

$$T(e_i) = e_1 \cdot T_{1i} + e_2 \cdot T_{2i} + \cdots + e_n \cdot T_{ni} \qquad T_{ij} \in \mathbb{C}$$

then $T(e_1, \cdots, e_n) = (e_1, \cdots, e_n) \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ T_{n1} & \cdots & \cdots & T_{nn} \end{pmatrix}$

$$\underbrace{\qquad\qquad\qquad}_{[T].}$$

↑ matrix.

$$T\cdot v = T\,(e_1,\cdots,e_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

$$= (e_1,\cdots,e_2)\underbrace{[T]\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}}_{\text{coefficients of }(Tv.)}$$

- It is in general, not meaningful to talk about eigenvalue & eigenvector of a matrix, unless the matrix comes from a linear transformation.