Week 9 (Lec 16 & 17)

24 October 2022 11:27

211. Find orthogonal bases and inertia indices of quadratic forms: ✓

$$x_1x_2 + x_2^2, \quad x_1^2 + 4x_1x_2 + 6x_2^2 - 12x_2x_3 + 18x_3^2, \quad x_1x_2 + x_2x_3 + x_3x_1.$$

(i) $x_1x_2 + x_2^2 = 0x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_1 + 1x_2^2 = Q$

$\Leftrightarrow (x_1 \ x_2)\begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let $B = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

$\tilde{B} = A^t \cdot B \cdot A$, where $\tilde{B}$ is a diagonal matrix

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix} \xrightarrow{C_1 \to C_1 - \frac{1}{2}C_2} \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \xrightarrow{r_1 \to r_1 - \frac{1}{2}r_2} \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$$

$$A: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{C_1 \to C_1 - \frac{1}{2}C_2} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$\Rightarrow$ Orthogonal bases are: $\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Inertia indices: $(p, q) = (1, 1)$

(ii) $Q = x_1^2 + 4x_1x_2 + 6x_2^2 - 12x_2x_3 + 18x_3^2$

$= 1x_1^2 + 2x_1x_2 + 2x_2x_1 + 6x_2^2 - 6x_2x_3 - 6x_3x_2 + 18x_3^2$

$\Leftrightarrow (x_1 \ x_2 \ x_3)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -6 \\ 0 & -6 & 18 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\Rightarrow$ let $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & -6 \\ 0 & -6 & 18 \end{bmatrix}$

$\tilde{B} = A^t \cdot B \cdot A$, where $\tilde{B}$ is a diagonal matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -6 \\ 0 & -6 & 18 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -6 \\ 0 & -6 & 18 \end{pmatrix} \xrightarrow{C_2 \to C_2 - 2C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -6 \\ 0 & -6 & 18 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -6 \\ 0 & -6 & 18 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -6 \\ 0 & -6 & 18 \end{pmatrix} \xrightarrow{C_2 \to C_2 - 2C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -6 \\ 0 & -6 & 18 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -6 \\ 0 & -6 & 18 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_3 \to C_3 + 3C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A^t: $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -6 & 3 & 1 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & -2 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ orthogonal bases are: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix} \right\}$

Inertia indices: $(p, q) = (2, 0)$

(iii) $Q = x_1 x_2 + x_2 x_3 + x_3 x_1$
$= 0x_1^2 + \frac{1}{2} x_1 x_2 + \frac{1}{2} x_2 x_1 + 0x_2^2 + \frac{1}{2} x_2 x_3 + \frac{1}{2} x_3 x_2 + 0x_3^2 + \frac{1}{2} x_3 x_1 + \frac{1}{2} x_1 x_3$

$$\Leftrightarrow (x_1 \ x_2 \ x_3) \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Rightarrow \text{Let } B = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$\tilde{B} = A^t . B . A$, where $\tilde{B}$ is a diagonal matrix

$$\begin{pmatrix} & 1/2 & 1/2 \\ & 0 & 1/2 \\ & 1/2 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{pmatrix} -\frac{1}{2} & 0 & 1/2 \\ \frac{1}{2} & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \xrightarrow{C_1 \to C_1 - C_3} \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ \frac{1}{2} & 1/2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} & 0 & 1/2 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 1/2 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{2}R_1} \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 \end{pmatrix} \xrightarrow{C_3 \to C_3 + \frac{1}{2}C_1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1/2 & 1/4 \end{pmatrix} \xrightarrow{C_2 \to C_2 - 2C_3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

$$A^t: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_3} \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 1 & 0 \\ -1 & 2 & \frac{1}{2} \end{pmatrix}$$

$\Rightarrow$ orthogonal bases are: $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \right\}$

Inertia indices : $(p, q) = (1, 2)$