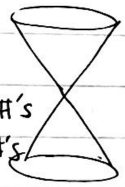
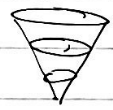
— Conic Curves

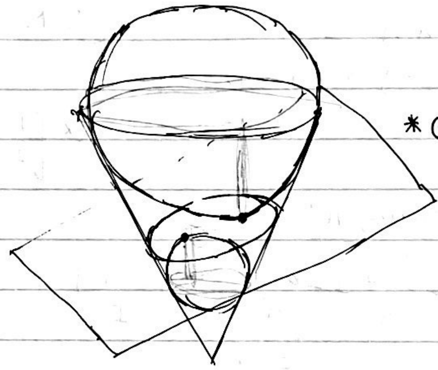
8/30/22

- $\mathbb{Z}$: integers
- $\mathbb{R}$: real #'s
- $\mathbb{Q}$: rational #'s
- $\mathbb{C}$: complex #'s



As "slope" increases,
ellipse → parabola → hyperbola





* Givental

Cone: $x_1^2 + x_2^2 = x_3^2$

Ex. $x^2 + y^2 = z^2$

- Expressing Conic Sections:

$$\left\{ (x_1, x_2, x_3) \;\middle|\; \begin{matrix} x_1^2 + x_2^2 = x_3^2 \\ a_1x_1 + a_2x_2 + a_3x_3 = a_0 \end{matrix} \right\}$$

* Solve by plugging $x_3 = \frac{a_0 - a_1x_1 - a_2x_2}{a_3}$ into eq. 1

Dfn: A conic curve in $\mathbb{R}^3$ is:

$$\{ (x_1, x_2) \in \mathbb{R}^2 \mid Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \}$$

* projection of intersection onto x_1-x_2 plane

$\boxed{\text{Circle}}$

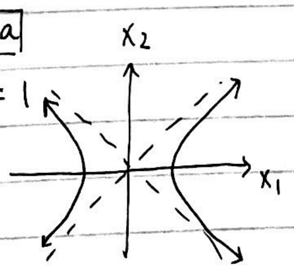
Ex. (1) $x_1^2 + x_2^2 = 1$ unit circle
$= r^2$

$\Leftrightarrow \left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 = 1$

(2) $\left(\frac{x_1}{r_1}\right)^2 + \left(\frac{x_2}{r_2}\right)^2 = 1$ ellipse

Ex. $\boxed{\text{Hyperbola}}$

$x_1^2 - x_2^2 = 1$



Ex. $\boxed{\text{Parabola}}$

$x_1^2 = x_2$

* Starting from the following 3:

$$x_1^2 + x_2^2 = 1$$
$$x_1^2 - x_2^2 = 1$$
$$x_2^2 = x_1^2$$

$\rightarrow$ using $\overbrace{\text{linear}}^{\text{"affine"}}$ transformations, we can obtain all possible conic curves

Vocab

T "transformation"

$\mapsto$ "maps to"

What's a linear transformation? * Doesn't include translation

Dfn. 1 $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

Def. 2 $T: \mathbb{R}^2 \to \mathbb{R}^2$, where

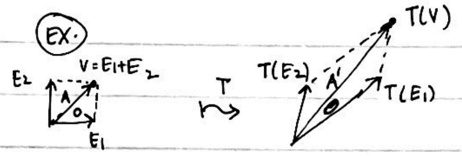
(1) $T(c \cdot \vec{v}) = c \cdot T(\vec{v}) \quad \forall c \in \mathbb{R}$

(2) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$

i.e. T (respect) preserves <u>linear structure</u> on both sides

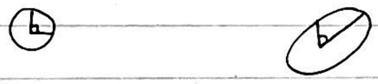
Prop 1: Fix a basis $\{E_1, E_2\}$ of $\mathbb{R}^2$. A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by how it acts on the basis vector.

EX:



$$T(V) = T(E_1 + E_2)$$
$$= T(E_1) + T(E_2)$$

Conceptual: You have a rubber sheet & are stretching & shrinking it w/ ~~tran~~ linear transformations



<u>Pf:</u> If any vector $v \in \mathbb{R}^2$, can be written as $V = a_1E_1 + a_2E_2$. (by def of the basis vectors)

- By linearity of T, $T(v) = T(a_1E_1 + a_2E_2)$
$$= T(a_1E_1) + T(a_2E_2)$$
$$= a_1T(E_1) + a_2T(E_2)$$

Hence T(v) is "determined".
- enough constraints

$$T(E_1) = a_{11}E_1 + a_{12}E_2$$
$$T(E_2) = a_{21}E_1 + a_{22}E_2,$$

then $T(c_1E_1 + c_2E_2) = \_\_\_ E_1 + \_\_\_ E_2$

$$= c_1T(E_1) + c_2T(E_2)$$
$$= c_1(a_{11}E_1 + a_{12}E_2) + c_2(a_{21}E_1 + a_{22}E_2)$$
$$= (c_1a_{11} + c_2a_{21})E_1 + (c_1a_{12} + c_2a_{22})E_2$$
$$= (c_1 \; c_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$\downarrow$

$$T\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = A\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \text{ where } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$T\left((c_1 \; c_2)\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}\right) = (c_1 \; c_2)\, T\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$
$$= (c_1 \; c_2)[A]\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

Matrix: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad (A \cdot B)_{ij} = \sum_{k=1}^{2} a_{ik}b_{kj}$

$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $\quad (A \cdot B)_{11} = a_{11}b_{11} + a_{12}b_{21}$

$$h\Big\{\overbrace{\begin{bmatrix} \; A \; \end{bmatrix}}^{l} \cdot l\Big\{\overbrace{\begin{bmatrix} \; B \; \end{bmatrix}}^{m} = h\overbrace{\begin{bmatrix} \quad \end{bmatrix}}^{m}$$

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik} \cdot B_{kj}$$