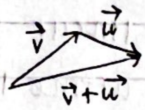
Re call: 9/1/22

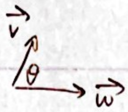
(1) Geometric Def. of a Vector:

- directed segment: $\overrightarrow{AB}$ 
- vector as dir. seg. upto translation

· Scalar Multiplication.

- addition: head to tail 

· Inner Product

$\langle \vec{v}\ \vec{w} \rangle = |\vec{v}|\,|\vec{w}| \cos\theta$ 

(2) Basis & Coordinates

- $\{E_1, E_2\}$ is a basis iff $\mathbb{R}^2$ for any $v \in \mathbb{R}^2$ there exist a unique way to write

$$V = C_1 E_1 + C_2 E_2$$

where (C_1, C_2) are called the coordinates of v. w/ respect to $(E_1 E_2)$

- Quadratic Form:

$Q(\vec{x}) = a x_1^2 + b x_1 x_2 + c x_2^2$
where $\vec{x} = (x_1, x_2)$

* homogeneous degree 2 polynomial in x_1, x_2 variables

· Given Q quadratic form we can try to solve $Q(\vec{x}) = 1$.

$\rightsquigarrow$ ellipse O
hyperbola)(
empty curve, ...

· Linear Transformations:

$T: \mathbb{R}^2 \to \mathbb{R}^2$

- Def 1: is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that preserves "linear structure"
  - scalar mult. & add.

· Prop 1: $T(\vec{0}) = \vec{0}$

a) $T(c\vec{0}) \stackrel{?}{=} cT(\vec{0})$
$T(\vec{0}) = c \cdot \vec{0}$
$\vec{0} = \vec{0}$

b) $T(\vec{a} + \vec{0}) \stackrel{?}{=} T(\vec{a}) + T(\vec{0})$
$T(\vec{a}) \stackrel{?}{=} T(\vec{a}) + \vec{0}$
$T(\vec{a}) = T(\vec{a})$

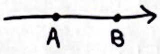
* prop. preserved when $T(\vec{0}) = \vec{0}$

$$\boxed{T(C_1 E_1 + C_2 E_2) = C_1 T(E_1) + C_2 T(E_2)}$$

- hence if we know $T(E_1)$ & $T(E_2)$, we know $T(v)$ for any $v \in \mathbb{R}^2$

Notation:
set:
- collection of objects

• **Affine Linear Transformation:**



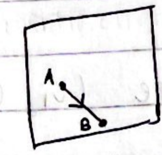
· $A+B=?$ not well-defined
· $A-B=?$ $\overrightarrow{BA}$, a directed segment
⇝ a vector

choose origin ↗ vector space = linear space ⤸ forget origin
affine space = affine linear space
· can do subtraction, not addition

Linear Space | **Affine Linear Space**

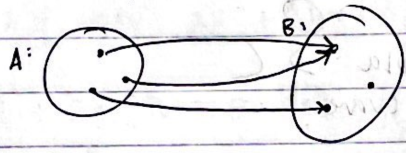
$\mathbb{R}^2$ $\qquad$ $-: \mathbb{A}^2 \times \mathbb{A}^2 \to \mathbb{R}^2$ $\qquad$ $\mathbb{A}^2$

$(A,B) \mapsto A-B = \overrightarrow{BA}$

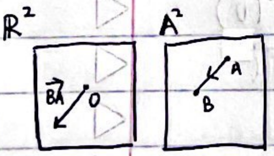


* vectors $\qquad$ * points

Aside on **Notation:**
• set: a collection of objects
(EX.) $\mathbb{N}, \mathbb{Z}, \mathbb{R}$
- If A and B are sets, a map $f: A \to B$ is an assignment of elements in A to elements in B.

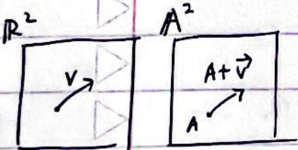


• Given two sets A, B, we can define "Cartesian product"
$A \times B = \{(a,b) \mid a \in A, b \in B\}$.
(EX) $A = \{1,2,3\}$
$B = \{a,b\}$
$A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$



"$-$" : $\mathbb{A}^2 \times \mathbb{A}^2 \to \mathbb{R}^2$ } declaration
subtraction $(A, B) \mapsto A-B = \overrightarrow{BA}$ } implementation

10"



"$+$" : $\mathbb{A}^2 \times \mathbb{R}^2 \to \mathbb{A}^2$
addition $(A, \vec{v}) \mapsto A+\vec{v}$

Affine Linear Transform. Cont...

$$\left[\begin{array}{l} T: \mathbb{A}^2 \to \mathbb{A}^2 \\ (x_1, x_2) \mapsto (\tilde{x}_1, \tilde{x}_2) \end{array}\right] \qquad \begin{array}{l} \tilde{x}_1 = a_{11}x_1 + a_{12}x_2 + \underset{\sim}{a_{10}} \\ \tilde{x}_2 = a_{21}x_1 + a_{22}x_2 + \underset{\sim}{a_{20}} \end{array}$$

represent freedom to move

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}$$

* Have already been working w/ them, moving vectors around

* Regular (not affine)

Prop: Any quadratic form

$$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$$

can be written in a new coordinate $(\tilde{x}_1, \tilde{x}_2)$ as one of the following forms:

① $\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{c}$ ①' $-\tilde{x}_1^2 - \tilde{x}_2^2 + \tilde{c}$

② $\tilde{x}_1^2 - \tilde{x}_2^2 + \tilde{c}$

③ $\tilde{x}_1^2 + \tilde{c}$

④ $0 + \tilde{c}$ where $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for some a_{ij} that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$

-Proof:

Assuming $a \neq 0$. $ax_1^2 + bx_1x_2 + cx_2^2$

$$= a\left[x_1^2 + \frac{b}{a}x_1x_2\right] + cx_2^2$$

$$= a\left[\left(x_1 + \frac{b}{2a}x_2\right)^2 - \frac{b^2}{4a^2}x_2^2\right] + cx_2^2$$

$$= a\left[x_1 + \frac{b}{2a}x_2\right]^2 - \frac{b^2}{4a}x_2^2 + cx_2^2$$

$$= a\left[x_1 + \frac{b}{2a}x_2\right]^2 + \left(c - \frac{b^2}{4a^2}\right)x_2^2$$

· case by case:

1) If $a > 0$, $c > \frac{b^2}{4a}$, then define:

$$\tilde{x}_1 = \sqrt{a} \cdot \left(x_1 + \frac{b}{2a}x_2\right)$$

$$\tilde{x}_2 = \sqrt{c - \frac{b^2}{4a}} \cdot x_2$$

2) If $a < 0$, $c < 0$, then define similarly

$$\tilde{x}_1 = \sqrt{|a|}\left(x_1 + \frac{b}{2a}x_2\right)$$

$$\tilde{x}_2 = \sqrt{\left|c - \frac{b^2}{4a^2}\right|}\, x_2$$

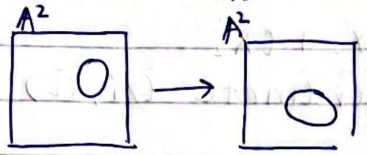
What if we had additional terms?

- $Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f$,

  then it could be written in a new affine linear coordinate $(\tilde{x}_1, \tilde{x}_2)$

  * can rotate basis to get rid of linear terms

Summary:

- Can't do addition in affine linear space anymore
- Why is aff. linear space important?
  - 

    - up to affine linear space, these two are the same
    - reduces classification