Math H54 Week 1

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1. Problem 6

   The first thing we want to do is represent the circles with equations with respect to time. Courtesy to Bryan Li, the equations are the following where (h_i, k_i) is the center and a_i is the radius:

$$r_1 =< h_1 + a_1 cos\omega t, k_1 + a_1 sin\omega t >$$
$$r_2 =< h_2 + a_2 cos\omega t, k_2 + a_2 sin\omega t >$$
$$r_3 =< h_3 + a_3 cos\omega t, k_3 + a_3 sin\omega t >$$

   It is stated that they have the same angular velocities, so their ω 's are the same.

   As Bryan also pointed out, the theorem given in problem 4, $\vec{OM} = \frac{1}{3}(\vec{OA} + \vec{OB} + \vec{OC})$ is crucial to solving the problem. The equation is defined where O is located at the origin $< 0, 0 >$, A is located on Circle 1, B is located on Circle 2, and C is located on Circle 3. Utilizing this information, we can define the following:

$$\vec{OA} =< h_1 + a_1 cos\omega t, k_1 + a_1 sin\omega t >$$
$$\vec{OB} =< h_2 + a_2 cos\omega t, k_2 + a_2 sin\omega t >$$
$$\vec{OC} =< h_3 + a_3 cos\omega t, k_3 + a_3 sin\omega t >$$
$$\vec{OM} = \frac{1}{3} < h_1 + h_2 + h_3 + (a_1 + a_2 + a_3)\cos\omega t, k_1 + k_2 + k_3 + (a_1 + a_2 + a_3)\sin\omega t >$$

   By observation we see that $\vec{OM}$ or the barycenter sweeps out a circle with center at $\frac{1}{3}(h_1 + h_2 + h_3, k_1 + k_2 + k_3)$ and radius $\frac{1}{3}(a_1 + a_2 + a_3)$ at the same angular velocity ω.

2. Problem 7

$$\vec{AA'} = \frac{1}{2}(\vec{AB} + \vec{AC})$$
$$2\vec{AA'} = \vec{AB} + \vec{AC}$$
$$(\vec{AA'} - \vec{AB}) + (\vec{AA'} - \vec{AC}) = \vec{0}$$
$$\vec{BA'} + \vec{CA'} = \vec{0}$$
$$\vec{0} = \vec{0}$$

   Explanation for the last statement: Because A' is the midpoint, $\vec{BA'}$ and $\vec{CA'}$ have the same magnitude. Furthermore, since they are opposite in orientation, they add up to the zero vector. Thus this statement is true.

3. Problem 8
   There exists a triangle ΔABC where A', B', and C' are its midpoints which are opposite correspondingly to the vertices A, B, and C. In order for the medians of ΔABC to form a triangle, they must add up to $\vec{0}$.

   Using the previous theorem in Problem 7, we can define $\vec{AA'}$, $\vec{BB'}$, and $\vec{CC'}$ in terms of two other sides:

$$\begin{aligned}\vec{AA'} &= \frac{1}{2}(\vec{AB}+\vec{AC})\\ \vec{BB'} &= \frac{1}{2}(\vec{BA}+\vec{BC})\\ \vec{CC'} &= \frac{1}{2}(\vec{CA}+\vec{CB})\end{aligned}$$

   Thus adding them up together:

$$\vec{AA'}+\vec{BB'}+\vec{CC'} = \frac{1}{2}(\vec{AB}+\vec{AC})+\frac{1}{2}(\vec{BA}+\vec{BC})+\frac{1}{2}(\vec{CA}+\vec{CB})$$

   And moving terms around:

$$\vec{AA'}+\vec{BB'}+\vec{CC'} = \frac{1}{2}(\vec{AB}+\vec{BA})+\frac{1}{2}(\vec{AC}+\vec{CA})+\frac{1}{2}(\vec{BC}+\vec{CB})$$

   Since they are opposite in direction, they cancel each other out when they're added together:

$$\begin{aligned}\vec{AA'}+\vec{BB'}+\vec{CC'} &= \vec{0}+\vec{0}+\vec{0}\\ &= \vec{0}\end{aligned}$$

   As a result, the medians of ΔABC form a triangle.

4. Problem 10
   By observing the diagram and utilizing the theorem from problem 7:

$$\begin{aligned}\vec{AA'} = \vec{DF} &= \frac{1}{2}(\vec{AB}+\vec{AC})\\ \vec{BB'} = \vec{FE} &= \frac{1}{2}(\vec{BC}+\vec{BA})\\ \vec{CC'} = \vec{ED} &= \frac{1}{2}(\vec{CB}+\vec{CA})\end{aligned}$$

   And then now expressing the medians of the second triangle with respect to the first:

$$\begin{aligned}\vec{DD'} &= \frac{1}{2}(\frac{1}{2}(\vec{AB}+\vec{AC})+\frac{1}{2}(\vec{BA}+\vec{CA}))\\ \vec{EE'} &= \frac{1}{2}(\frac{1}{2}(\vec{CB}+\vec{CA})+\frac{1}{2}(\vec{CB}+\vec{AB}))\\ \vec{FF'} &= \frac{1}{2}(\frac{1}{2}(\vec{BA}+\vec{CA})+\frac{1}{2}(\vec{BC}+\vec{BA}))\end{aligned}$$

   Combining the terms, the following is obtained:

$$\begin{aligned}\vec{DD'} &= \frac{1}{4}(2\vec{AC}+\vec{BC}+\vec{AB})\\ \vec{EE'} &= \frac{1}{4}(2\vec{CB}+\vec{CA}+\vec{AB})\\ \vec{FF'} &= \frac{1}{4}(2\vec{BA}+\vec{BC}+\vec{CA})\end{aligned}$$

Then something convenient appears! The rightmost two terms add up to the first term:

$$\vec{DD'} = \frac{1}{4}(2\vec{AC} + \vec{AC})$$
$$\vec{EE'} = \frac{1}{4}(2\vec{CB} + \vec{CB})$$
$$\vec{FF'} = \frac{1}{4}(2\vec{BA} + \vec{BA})$$

Further simplifying:

$$\vec{DD'} = \frac{3}{4}(\vec{AC})$$
$$\vec{EE'} = \frac{3}{4}(\vec{CB})$$
$$\vec{FF'} = \frac{3}{4}(\vec{BA})$$

The third triangle is composed of the medians of the second triangle or correspondingly $\vec{DD'}$, $\vec{EE'}$, and $\vec{FF'}$. Because the sides of the third triangle are simply a scalar multiple of the original sides of the triangle, it is similar to the first triangle. And seen through the above equations, the coefficient of similarity is $\frac{3}{4}$.

5. Problem 11

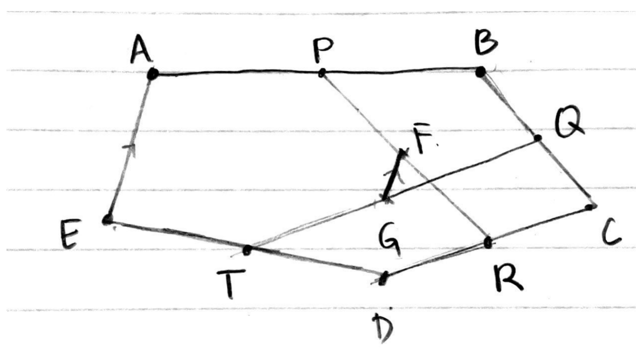

Figure 1: Fig. 1

$$\vec{OP} = \frac{1}{2}(\vec{OA} + \vec{OB})$$
$$\vec{OR} = \frac{1}{2}(\vec{OD} + \vec{OC})$$
$$\vec{OT} = \frac{1}{2}(\vec{OE} + \vec{OD})$$
$$\vec{OQ} = \frac{1}{2}(\vec{OB} + \vec{OC})$$

Using the theorem from problem 7:

$$\begin{aligned}\vec{OF} &= \frac{1}{2}(\vec{OP} + \vec{OR}) \\ &= \frac{1}{4}(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}) \\ \vec{OG} &= \frac{1}{2}(\vec{OT} + \vec{OQ}) \\ &= \frac{1}{4}(\vec{OE} + \vec{OD} + \vec{OB} + \vec{OC})\end{aligned}$$

Thus when we subtract, many of the vectors cancel out, giving:

$$\vec{OG} - \vec{OF} = \frac{1}{4}(\vec{OE} - \vec{OA})$$

$$\vec{FG} = \frac{1}{4}\vec{EA}$$

Finally, we see that the segment is 1/4 in length of $\vec{EA}$ and since it's a scalar multiple of $\vec{EA}$, it's parallel to it as well.

6. Problem 12

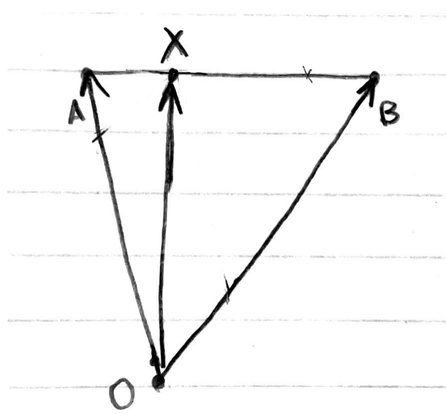

Figure 2: Fig. 2

$$\begin{aligned}\vec{OX} &= \vec{OB} + \lambda\vec{BA} \\ &= \vec{OB} + \lambda(\vec{OA} - \vec{OB}) \\ &= \lambda\vec{OA} + (1-\lambda)\vec{OB}\end{aligned}$$