Disclaimer: In no way, shape, or form am I claiming to know or understand anything about linear algebra. Nothing has changed between last week and this week.

This is clarification regarding the problem in discussion section where we were supposed to transform $x^2 - 2xy + y^2 = 1$ into $AX^2 + BY^2 = C$ because I didn't understand what our GSI did. So... this is basically for me to explain my thoughts... bc instead of working on hw, I spent time understanding this smh.

Let's get started.

What is orthogonal diagonalization?

According to Wikipedia, it is diagonalizing a matrix by doing an orthogonal change of coordinates (paraphrased from the Wikipedia page on orthogonal diagonalization). According to a much simpler explanation on Stack Exchange, it is where the columns of one matrix P are an orthonormal basis and a basis of eigenvectors for another matrix A. (If you don't know what eigenvectors are, look it up. I didn't know either.)

In order to do orthogonal diagonalization for a $n \times n$ matrix (a square matrix) P:

1) P is invertible and $P^{-1} = P^T$.
2) The rows of P are orthonormal.
3) The columns of P are orthonormal.

Okay, before we do anything... let's look at our basis vectors for $x = y$.

According to the problem in the textbook (ODE), "the quadratic form xy is symmetric about $x = y$" ("ODE", 11). So, we are going to want to start off with the basis vectors for $x = y$. (I used the pic in link 1 under this section on my page within the title "Useful Links to help visualize why the vectors became like they did...).

So... the basis vectors for x = y are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$.

Remark:

2 x 2 matrix is invertible if determinant does not equal 0.

Determinant - in which you calculate the determinant by taking matrix A, where A is equal to $\begin{bmatrix}a & b\\c & d\end{bmatrix}$, and doing: $ad - bc$.

So... if we were to calculate the determinant... we would use our basis vectors for $x = y$, which are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$.

So, matrix A:

$A = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

To calculate our determinant, we will take: $(1 * -1) - (1 * 1) = 2$. 2 is not equal to 0, which means that our matrix A is invertible, satisfying the first part of the first condition.

Now, to check if the second part of the first condition is true, we need to know that $P^{-1} = P^T$ is the same as $PP^T = I$, where P^T is the transpose of P and where I is the identity matrix.

The **transpose matrix** is basically a matrix with its rows and columns switched. For instance, the transpose of matrix B $\begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix}$ is $\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$.

***So, the transpose of matrix** A **is**:* $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

The **identity matrix** is a square matrix with diagonal entries (from upper left to lower right) of 1's and all other entries of 0's. For instance: I_2: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, I_3: $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$, I_4: $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$, and so on.

Note: The product of any matrix and the identity matrix is always the original matrix!

So, let's check if $AA^T = I$, where matrix A is the basis vector of $x = y$.

This consists of matrix multiplication, where ith rows of matrix A is multiplied by jth columns of matrix A^T.

$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}1\times 1 + 1\times 1 & 1\times 1 + 1\times -1\\1\times 1 + 1\times -1 & 1\times 1 + -1\times -1\end{bmatrix}$, resulting in this matrix: $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$, which is evidently not the identity matrix.

Since these vectors do not satisfy the condition for orthogonal diagonalization, we can normalize the vectors, which essentially is converting these vectors to the unit vectors of the line $x = y$ to create an orthonormal basis that will satsify these conditions.

To normalize a matrix, simply calculate the magnitude of the matrix and and multiply the magnitude by the vectors.

For this matrix, the rows are orthogonal (we know this by calculating the dot product of each separate vector and getting the result 0). The columns are not orthogonal (since AA^T does not equal 0). So, we need to normalize the rows.

Calculate the magnitude of each row in matrix A, where we get, for the first row $\sqrt{1^2+1^2} = \sqrt{2}$. For the magnitude for the second row, we also get $\sqrt{1^2+(-1)^2} = \sqrt{2}$. Now, divide each row by its magnitude. By doing this, we get the matrix A':

$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$.

To check if A' is now orthogonal, let's use matrix multiplication, multiplying A' by A'^T, where A'^T is equal to

$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$

to see if it gives us the identity matrix:

$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1/2+1/2 & 1/2+(-1/2)\\1/2-1/2 & 1/2+1/2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I.$

So, now we know the new matrix A' is orthonormal.

And we are almost done! Since the new unit vectors are $f_1 = (1/\sqrt{2}, 1/\sqrt{2})$ and $f_2 = (1/\sqrt{2}, -1/\sqrt{2})$, we can scale the coordinates $x_1^{'}$ and $x_2^{'}$ by $1/\sqrt{2}$, where:

$x_1 = x + y$

$x_1^{'} = 1/\sqrt{2}(x+y) = (x+y)/\sqrt{2}$

and

$x_2 = x - y$

$x_2^{'} = 1/\sqrt{2}(x-y) = (x-y)/\sqrt{2}$

So, that's how we ended up with the equation for the ellipse, where the quadratic form xy is equal to:

$(x+y)^2/(\sqrt{2})^2$ - $(x-y)^2/(\sqrt{2})^2 = X_1^2/2 - X_1^2/2.$