Recall:

Def: (basis) A subset B of a vector space V is a basis if every $v \in V$ can be written uniquely as a finite linear combination of vectors in B

Prop: B is a basis $\Leftrightarrow$ B is linearly independent & B spans V

Pf: If B is not lin. indep. then $\exists$ $a_1 b_1 + \cdots + a_n b_n = 0$ $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$

then say $a_1 \neq 0$, then $b_1 = -\frac{1}{a_1}(a_2 b_2 + \cdots + a_n b_n)$

contradicting that every vector can be written uniquely as finite linear combination of B

Def: V is finite dimensional, if there exists a a finite subset $\{v_1, \ldots, v_m\}$ that spans V

Lemma, if V is finite dimensional, then there exist a basis B of finitely many vectors

A vector $v \in S \subset V$ is redundant if $\operatorname{span} S = \operatorname{span} S \setminus \{v\}$

in other word, $v \in \operatorname{span}(S \setminus \{v\})$

Let $B = \{v_1, \ldots, v_n\}$ be a basis of V

Let $\Phi : K^n \longrightarrow V$ be given by

$$(x_1, \ldots, x_n) \longmapsto x_1 v_1 + \cdots + x_n v_n$$

Then Φ is an isomorphism of v.s.
$\parallel$
bijective linear map

Prove there is no isomorphism between $\mathbb{K}^2$ and $\mathbb{K}^3$

Lemma: any $n+1$ vectors in $\mathbb{K}^n$ are linearly dependent

For $n=1$, the statement says any 2 vectors in $\mathbb{K}$ are lin. dep.

If one of the vectors is 0, then $\{v_1, v_2\}$ in lin dep.

If v_1 v_2 both nonzero in $\mathbb{K}$, then $\frac{v_1}{v_2} \cdot v_2 - v_1 = 0$

· Suppose the statement is true for $n = 1, 2, \cdots, n_0 - 1$.
we need to show the case $n = n_0$
Let $\{v_1, \cdots, v_{n+1}\}$ be $n+1$ vectors in K^n
write them as column vectors

$$\begin{array}{cccc} v_1 & v_2 & \cdots & v_{n+1} \end{array}$$
$$\begin{pmatrix} v_{1} & & & v_{n+1,1} \\ v_{2} & & & v_{n+1,2} \\ \vdots & & & \vdots \\ v_{n} & & & v_{n+1,n} \end{pmatrix}$$

· If all the last coordinates of v_i are zero, then
$v_1, \cdots, v_{n+1} \in K^{n-1}$
by induction, we know ?
$\{v_1, \cdots, v_n\}$ are linearly in
· if not, by rearranging the ordering of column, we may assume $v_{n+1,n} \neq 0$
then consider $\tilde{v}_1 = v_1 - \frac{v_{1n}}{v_{n+1,n}} \cdot v_{n+1}$

$\vdots$

$\tilde{v}_n = v_n - \frac{v_{nn}}{v_{n+1,n}} \cdot v_{n+1}$

thus $\tilde{v}_{in} = v_{in} - \frac{v_{in}}{v_{n+1,n}} v_{n+1,n} = 0$

Thus $\{\tilde{v}_1, \cdots, \tilde{v}_n)$ are in K^{n-1}

hence by induction, these vectors are linearly dependent.

Thus $\exists\ (a_1, \cdots, a_n) \in K^n$ non zero (none of a_i is) s.t. $a_1\tilde{v}_1 + \cdots + a_n\tilde{v}_n = 0$

Thus $a_1(v_1 - c_1 v_{n+1}) + \cdots + a_n(v_n - c_n \cdot v_{n+1}) = 0$

$\Leftrightarrow a_1 v_1 + \cdots + a_n v_n + (-a_1 c_1 \cdots - a_n c_n) v_{n+1} = 0$

thus $v_1, \cdots, v_{n+1}$ are linearly dep

Ex. $\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$ $\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 3 \end{pmatrix}$ to show v_1, v_2, v_3 in $\mathbb{R}^2$ are lin. dep.

$\tilde{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$

$\tilde{v}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \frac{5}{3} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ 0 \end{pmatrix}$ $\quad 7\tilde{v}_1 - \tilde{v}_2 = 0$

Cor: (1) for any $m > n$ any m vectors in $\mathbb{K}^n$ are linearly dependent

(2) If $\exists$ an isomorphism $T : \mathbb{K}^n \to \mathbb{K}^m$ then $n = m$

Pf: if not, say $n > m$, then $\{e_1, \cdots e_n\}$ basis in $\mathbb{K}^n$ is sent to $Te_1, \cdots, Te_n$ in $\mathbb{K}^m$

since $n > m$, by the lemma

$\exists$ a linear combination

$$a_1(Te_1) + \cdots + a_n(Te_n) = 0 \quad \text{in } \mathbb{K}^m$$

where one of the $a_i \neq 0$

But T is invertible, apply T^{-1} to both sides

$$a_1 e_1 + \cdots + a_n e_n = 0 \text{ in } \mathbb{K}^n \text{ where some } a_i \neq 0$$

but this is impossible since $e_1, \ldots, e_n$ are lin. indep. in $\mathbb{K}^n$

(3) for any fin. dim. vector space V, there is a unique n, s.t. there is an isomorphism

$$T : \mathbb{K}^n \to V$$

Pf: existence of $T : \mathbb{K}^n \to V$ is guaranteed by existence of basis, n here $=: \dim_{\mathbb{K}} V$

if $m \neq n$ we know there is no isom from $\mathbb{K}^m \to V$

(otherwise, if exists isom $S : \mathbb{K}^m \to V$, then $\mathbb{K}^m \xrightarrow{S} V \xrightarrow{T^{-1}} \mathbb{K}^n$ is an isom. which contradicts (2))

(4) if 2 vector spaces V, W have the same dimension, then $\exists$ isom $T : V \to W$

Pf:
$$\begin{array}{ccc} T_1 : \mathbb{K}^n & \xrightarrow{\sim} & V \\ \downarrow \cong & & \\ T_2 \quad \mathbb{K}^n & \xrightarrow{\sim} & W \end{array}$$

concretely, pick basis $\{v_1, \ldots, v_n\}$ of V $\{w_1, \ldots, w_n\}$ of W

define T s.t. $T(v_i) = w_i$

$=$ equal

$\simeq$ isom.

$\xrightarrow{\sim}$ isom.

$\xrightarrow{\simeq}$ with direction

Thm (rank)

Let $T: V \to W$ be a lin map of fin. dim. v.s.

Let $r = \text{rank}(T) := \dim \text{im}(T)$

Then $\exists$ a basis $\{v_1, \cdots, v_n\}$ of V and a basis $\{w_1, \cdots, w_m\}$ of W such that

$$T(v_1) = w_1$$
$$\vdots$$
$$T(v_r) = w_r$$
$$T(v_{r+1}) = 0$$
$$T(v_n) = 0$$

Pf:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & W \\ V/\ker T \arrow[r, "\sim"] & (\text{im} T) \arrow[u, hook] \end{tikzcd}$$

find a basis $\{w_1, \cdots, w_r\}$ for $\text{im} T$

For any $w \in \text{im} T$, $T^{-1}(w)$ is an equivalence class of $V/\ker T$

complete the basis $\{w_1, \cdots, w_r\}$ of $\text{im} T$ $\{w_1, \cdots, w_m\}$ a basis of W

pick $v_1 \in T^{-1}(w_1), \cdots, v_r \in T^{-1}(w_r)$

Let $\{v_{r+1}, \cdots, v_n\}$ be the basis for $\ker T$

claim: $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ form a basis of V

For any $v \in V$, consider its image $[v] \in V/\ker T$

then $[v] = a_1[v_1] + \cdots + a_r[v_r]$ in $V/\ker T$ for some a_i

Thus $v - (a_1 v_1 + \cdots + a_r v_r) \in \ker T$

since $\{v_{r+1}, \cdots, v_n\}$ is a basis of $\ker T$, we know $\exists!$ $a_{r+1} \cdots a_n$ such that $v - (a_1 v_1 + \cdots + a_r v_r) = a_{r+1} v_{r+1} + \cdots + a_n v_n$

Thus $v = a_1 v_1 + \cdots + a_n v_n$

Cor. If V, V' of $\dim n$, W, W' of $\dim m$ and $T: V \to W$ $T': V' \to W'$ of rank r then $\exists$ isom $V \simeq V'$, $W \simeq W'$

such that
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "A"', "\simeq"] & W \arrow[d, "B"', "\simeq"] \\ V' \arrow[r, "T'"] & W' \end{tikzcd}$$

such that $T = (B)^{-1} \circ T' \circ A$

$$E_r = \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} & 0 \\ \hline 0 & 0 \end{array}\right)$$

Pf:
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\simeq"] & W \arrow[d, "\simeq"] \\ K^n \arrow[r, "E_r"] & K^m \\ V' \arrow[u, "\simeq"] \arrow[r, "T'"] & W' \arrow[u, "\simeq"] \end{tikzcd}$$