HW 4:

86.

$$BAC=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&2&3\\1&-1&1\end{bmatrix}\begin{bmatrix}1&2\\-2&1\\0&-1\end{bmatrix}=\begin{bmatrix}-1&2\\3&4\end{bmatrix}\begin{bmatrix}-3&1\\3&0\end{bmatrix}=\begin{bmatrix}3&1\\3&3\end{bmatrix}$$

$$CBA=\begin{bmatrix}1&2\\-2&1\\0&-1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&2&3\\1&-1&1\end{bmatrix}=\begin{bmatrix}7&10\\1&0\\-3&-4\end{bmatrix}\begin{bmatrix}1&2&3\\1&-1&1\end{bmatrix}=\begin{bmatrix}17&4&31\\1&2&3\\-7&-2&-13\end{bmatrix}$$

$$ACB=\begin{bmatrix}1&2&3\\1&-1&1\end{bmatrix}\begin{bmatrix}1&2\\-2&1\\0&-1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}-3&1\\3&0\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}0&-2\\3&6\end{bmatrix}$$

$A_{ii}, B_{ij}=0$ for $i>j$

88. Let A, B be two upper triangular matrices with the same size $n\times n$

$$[AB]_{ij}=\sum_{k=1}^{n}A_{ik}B_{kj} \quad [AB]_{ij}=\underbrace{\sum_{k=1}^{i-1}a_{ik}b_{kj}}_{\Uparrow\ a_{ik}=0}+\sum_{k=i}^{n}a_{ik}b_{kj}$$

$$=\sum_{k=i}^{n}a_{ik}b_{kj}$$

If $i>j$, $b_{kj}=0$ $(k>i \Leftrightarrow k>j)$, $\sum_{k=i}^{n}a_{ik}b_{kj}=0$

$\therefore [AB]_{ij}=0$ for $i>j$

91. $\begin{bmatrix}\cos 19 & -\sin 19\\ \sin 19 & \cos 19\end{bmatrix}^{19}$ is rotating counter-clockwise 19° by 19 times, which is equal to rotating counter-clockwise 1°.

$$\therefore \begin{bmatrix}\cos 19^\circ & -\sin 19^\circ\\ \sin 19^\circ & \cos 19^\circ\end{bmatrix}^{19}=\begin{bmatrix}\cos 1^\circ & -\sin 1^\circ\\ \sin 1^\circ & \cos 1^\circ\end{bmatrix}$$

95. $(A+B)^2=A^2+AB+BA+B^2$

$AB=BA$ iff $A=B$ or A, B are diagonal matrices

98. $(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}=AA^{-1}=I$

$\therefore B^{-1}A^{-1}=(AB)^{-1}$

99. Yes $(AB)^{-1}=B^{-1}A^{-1}$ $\Leftarrow$ A, B are invertible

$A^{-1}B^{-1}=(BA)^{-1}$ $\Leftrightarrow$ BA is invertible

100. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

107. $(AB)^T = B^T A^T = BA \neq AB$

111. $\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$\begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1$

$\begin{vmatrix} \cos x & \sin y \\ \sin x & \cos y \end{vmatrix} = \cos x \cos y - \sin x \sin y = \cos(x+y)$

112. $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1 \times (-1) + 1 \times 1 = 2$

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = -1 \times (1 \times 6 - 1 \times 3) + 1 \times (1 \times 3 - 1 \times 2) = -3 + 1 = -2$

$\begin{vmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{vmatrix} = 1 \times 1 - i \times (-i) + (1+i) \times (i-1)$
$= 1 + i + i^2 - 1 = i^2 + i$

114. $l(\sigma) = 1 + 2 + 3 + \cdots + k-1 = \frac{k(k-1)}{2}$

$$\begin{pmatrix} 1 & 2 & & \cdots & k & k+1 & \cdots & 2k \\ 1 & 3 & 5 & \cdots & 2k-1 & 2\ 4 & \cdots & 2k \end{pmatrix}$$

$\uparrow$ 1 — # on the right less than 3

$\uparrow$ 2 — # on the right less than 5

116.

$$\begin{array}{ccc} 1 & i_1 & i_n \\ 2 & i_2 & i_{n-1} \\ 3 & i_3 & i_{n-2} \\ \vdots & \vdots & \vdots \\ n & i_n & i_1 \end{array}$$

$l(\sigma_1) = l$ $\quad l(\sigma_2) = \frac{n(n-1)}{2}$

Elementary swaps in σ_2 will undo swaps in σ_1.
The resultant $l(\sigma)$ is then $\frac{n(n-1)}{2} - l$

117. Given a permutation σ with length l
It takes l elementary transpositions to turn
σ into identity permutation

$$\underset{l}{\sigma}\,\underset{-1}{T^{(i1)}}\,\underset{-1}{T^{(i2)}}\,\underset{-1}{T^{(i3)}}\cdots\underset{-1}{T^{(il)}} = \underset{0}{id}$$

$$\Updownarrow$$

$$\underset{0}{(id)}\,\underset{+1}{T^{(il)}}\,\underset{+1}{T^{(il-1)}}\cdots\underset{+1}{T^{(i1)}} = \sigma^{-1}$$

$\therefore l(\sigma^{-1}) = l(\sigma) = l$

119. $\sigma = T^{(14)}\,T^{(25)}\,T^{(31)}$