HW 5:

110. $\det M = \sum_{\sigma} (-1)^{l(\sigma)} M_{1\sigma(1)} M_{2\sigma(2)} M_{3\sigma(3)} M_{4\sigma(4)} M_{5\sigma(5)}$

If $M_{1\sigma(1)}, M_{1\sigma(2)} \neq 0$, $\sigma(1), \sigma(2) \in \{4, 5\}$

$\Rightarrow \sigma(3) \in \{1, 2, 3\} \Rightarrow M_{3\sigma(3)} = 0 \Rightarrow \det M = 0$

118. $a=1, g=2, h=3, i=4, l=5, m=6, o=7, r=8, t=9$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 7 & 2 & 1 & 8 & 4 & 9 & 3 & 6 \end{pmatrix}$$

logarithm:

$l(\sigma) = 4+5+1+3+1+2 = 16$

$\varepsilon(\sigma) = (-1)^{l(\sigma)} = (-1)^{16} = 1$ (even)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 2 & 7 & 8 & 4 & 9 & 3 & 6 \end{pmatrix}$$

algorithm

$l(\sigma) = 3+3+3+1+2 = 12$

$\varepsilon(\sigma) = (-1)^{l(\sigma)} = (-1)^{12} = 1$ (even)

132. (a) $\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \end{vmatrix}$

$= 1 - 4 + 2 \times (-2) \times (-3) = 9$

(b) $\begin{vmatrix} -2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -2 \times \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$

$= -2 \times [2 \times 3 + 1 \times (-2)] + 1 \times (-1 \times 3)$

$= -8 - 3 = -11$

135. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

138. (a) $\begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}$

$\det A = 60$

$x_1 = \frac{180}{60} = 3$

$x_2 = \frac{60}{60} = 1$

$x_3 = \frac{60}{60} = 1$

138. (b) $\begin{pmatrix} 1 & 2 & 4 \\ 5 & 1 & 2 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 31 \\ 29 \\ 10 \end{pmatrix}$ $x_1 = \frac{-81}{-27} = 3$

$x_2 = \frac{-108}{-27} = 4$

$\det A = -27$

$x_3 = \frac{-135}{-27} = 5$

139. (a)

$n=1$, $\begin{vmatrix} 0 & x_1 \\ x_1 & 1 \end{vmatrix} = 0 - x_1^2 = -x_1^2$

$n=2$, $\begin{vmatrix} 0 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{vmatrix} = -x_1^2 - x_2^2$

$n=3$, $\det = -x_1^2 - x_2^2 - x_3^2$

$n=4$, $\det = -x_1^2 - x_2^2 - x_3^2 - x_4^2$

$\vdots$

$n=n$, $\det = -x_1^2 - x_2^2 - x_3^2 \cdots - x_n^2$

(b) $A = \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$ $B = \begin{vmatrix} 0 & 0 & b \\ 0 & b & 0 \\ b & 0 & 0 \end{vmatrix}$

$C = \begin{vmatrix} 0 & 0 & c \\ 0 & c & 0 \\ c & 0 & 0 \end{vmatrix}$ $D = \begin{vmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{vmatrix}$

$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(A - BD^{-1}C)\det(D)$

$A - BD^{-1}C = \begin{vmatrix} \frac{ad-bc}{d} & 0 & 0 \\ 0 & \frac{ad-bc}{d} & 0 \\ 0 & 0 & \frac{ad-bc}{d} \end{vmatrix}$

$\det(A - BD^{-1}C)\det(D)$

$= \left(\frac{ad-bc}{d}\right)^3 \cdot d^3 = (ad-bc)^3$

140. $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{vmatrix}$

$P_{12}P_{34} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})(a_{13}a_{24} - a_{14}a_{23})$

$P_{13}P_{24} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} = (a_{11}a_{23} - a_{13}a_{21})(a_{12}a_{24} - a_{22}a_{14})$

$P_{14}P_{23} = \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{11}a_{24} - a_{14}a_{21})(a_{12}a_{23} - a_{22}a_{13})$

$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = \cancel{a_{11}a_{22}a_{13}a_{24}} - \cancel{a_{11}a_{22}a_{14}a_{23}} - \cancel{a_{12}a_{21}a_{13}a_{24}}$
$+ \cancel{a_{12}a_{21}a_{14}a_{23}} - \cancel{a_{11}a_{23}a_{12}a_{24}} + \cancel{a_{11}a_{23}a_{22}a_{14}} + \cancel{a_{13}a_{21}a_{12}a_{24}} - \cancel{a_{13}a_{21}a_{22}a_{14}}$
$+ \cancel{a_{11}a_{24}a_{12}a_{23}} - \cancel{a_{11}a_{24}a_{22}a_{13}} - \cancel{a_{14}a_{21}a_{12}a_{23}} + \cancel{a_{14}a_{21}a_{22}a_{13}} = 0$

143. Cofactor Expansion

$$\det = (-1)^{n+1} a_n (-1)^{n-1} + (-1)^{n+2} a_{n-1} \cdot \lambda \cdot (-1)^{n-2}$$
$$+ (-1)^{n+3} a_{n-2} \cdot \lambda^2 \cdot (-1)^{n-3} + \cdots + (-1)^{n+n} \cdot (\lambda + a_1) \cdot \lambda^{n-1}$$
$$= \lambda^n + \lambda^{n-1} a_1 + \lambda^{n-2} a_2 + \cdots + \lambda^2 a_{n-2} + \lambda a_{n-1} + a_n$$