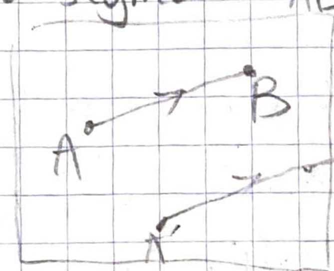
Linear Algebra

· vector:

(1) $\nearrow$ something with magnitude & direction
(2) $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ a column of numbers
(3) an element of a vector space
(4) an ordered collection of magnitudes

* directed segment: $\overrightarrow{AB}$

plane R^2
$\overrightarrow{AB}$ directed seg.
$[\overrightarrow{AB}]$ the vector it represents



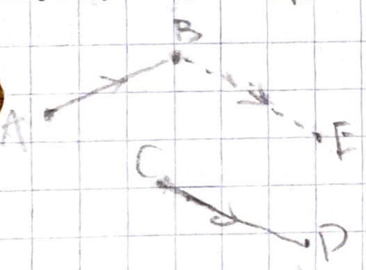
* translation: move the endpoints without changing the length & direction

. vector is an equivalence class of directed segment modulo translation
if 2 dir. seg differ by a translation, they represent the same vector

$\overrightarrow{AB}$ & $\overrightarrow{A'B'}$ are different dir. seg. but the same vector

· addition: $[\overrightarrow{AB}]$ $[\overrightarrow{CD}]$ two vectors ($\vec{u}$, $\vec{v}$)

* $\overrightarrow{MA} = \overrightarrow{OA} - \overrightarrow{OM}$



$[\overrightarrow{AB}] + [\overrightarrow{CD}]$
$= [\overrightarrow{AB}] + [\overrightarrow{BE}] = [\overrightarrow{AE}]$

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

One can do linear combination of 2 vectors:

· scalar multiplication

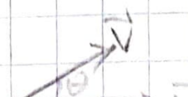
if $\vec{v}$, $\vec{w}$ are vectors and $a, b \in R$

$3 \cdot \vec{v}$
$= \vec{v} + \vec{v} + \vec{v}$

then $a \cdot \vec{v} + b \cdot \vec{w}$ is well defined

· inner product (dot product): $\langle \vec{v}, \vec{u} \rangle = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$

given 2 vectors, $\vec{v}$, $\vec{u}$
we define $\langle \vec{v}, \vec{u} \rangle \in R$



inner product is a linear operation
. linear means the structure of scalar multiplication & addition

$\langle \vec{v} + \vec{u}, \vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle + \langle \vec{u}, \vec{w} \rangle$
$\langle 3\vec{v}, \vec{w} \rangle = 3 \cdot \langle \vec{v}, \vec{w} \rangle$

[vector space: solution to $f''(x) = f(x)$]

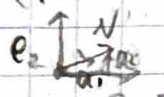
- $\langle \vec{u}, \vec{u} \rangle = |u|^2$

$$\langle u_1 + u_2, v_1 + v_2 \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle + \langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle$$

$$\cos\theta_{u,v} = \frac{\langle u, v \rangle}{\langle u, u \rangle^{\frac{1}{2}} \langle v, v \rangle^{\frac{1}{2}}}$$

Basis Vector:

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then any other vector in R^2 can be written as linear combination

$$v = \underline{a_1}\, e_1 + \underline{a_2}\, e_2$$

- e_1, e_2 forms an orthonormal basis

(orthogonal), $\langle e_1, e_2 \rangle = 0$, $e_1 \perp e_2$

normal, $|e_1| = 1$, $|e_2| = 1$

Claim: $v = \underline{\langle v, e_1 \rangle} e_1 + \langle \underline{v, e_2} \rangle e_2$

proof: suppose $v = a_1 e_1 + a_2 e_2$, then we apply $\langle \cdot, e_1 \rangle$ to both sides:

$$\langle v, e_1 \rangle = \langle a_1 e_1 + a_2 e_2, e_1 \rangle = a_1 \langle e_1, e_1 \rangle + a_2 \langle e_2, e_1 \rangle$$
$$= a_1 \cdot 1 + a_2 \cdot 0 = a_1$$

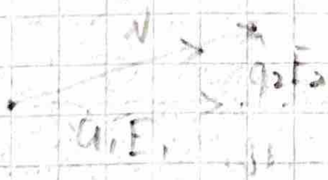
"skewed" basis on R^2

↑ not orthonormal

In general, if two vectors E_1, E_2 in $\mathbb{R}^2$ satisfies the property that, any vector $v \in R^2$ can be written as

$$v = a_1 E_1 + a_2 E_2$$

then we say (E_1, E_2) is a basis

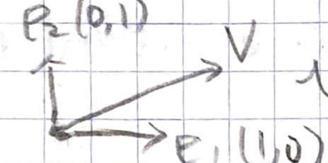


Coordinates:

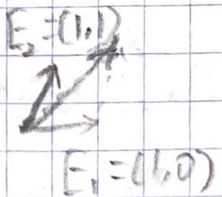
Given a basis E_1, E_2, we say a vector v has coordinate (a_1, a_2) if $v = a_1 E_1 + a_2 E_2$

Note: if you change basis, the coordinate will change

Ex.



$v = 2 \cdot e_1 + 1 \cdot e_2$



$v = 1E_1 + 1 \cdot E_2$

Conic Sections

As the "slope" of the plane increases, we get ellipse → parabola, hyperbola

ellipse

focal points fixed

ellipse - points on $\mathbb{R}^2$ such that $l_1 + l_2 = L$

A cone in $\mathbb{R}^3$: $x_1^2 + x_2^2 = x_3^2$

A line in $\mathbb{R}^2$ can be described by an equation $a_1 x_1 + a_2 x_2 = a_3$

A plane in $\mathbb{R}^3$: $a_1 x_1 + a_2 x_2 + a_3 x_3 = a_0$

A conic section is given by $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = x_3^2,\ a_1x_1 + a_2x_2 + a_3x_3 = a_0\}$

if $a_3 \neq 0$, then $x_3 = \dfrac{a_0 - a_1 x_1 - a_2 x_2}{a_3}$

plug in $x_3 = \cdots$ to the first eq.

$$x_1^2 + x_2^2 = \left(\frac{a_0 - a_1x_1 - a_2x_2}{a_3}\right)^2$$

$$\Leftrightarrow Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0$$ (projected to a plane)

$\{(x_1, x_2, x_3) \mid a_1x_1 + a_2x_2 + a_3x_3 = a_0\}$ is a vertical plane

Def: a conic curve in $\mathbb{R}^2$ is

$\{(x_1, x_2) \in \mathbb{R} \mid Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0\}$ A, B, C aren't 0 simultaneously

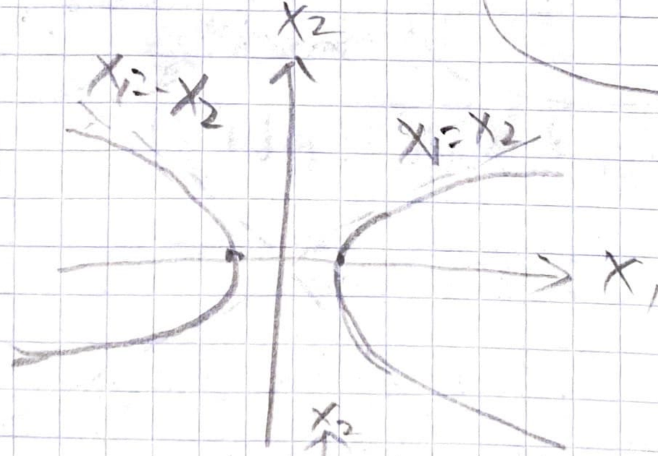
Ex: (1) $x_1^2 + x_2^2 = 1$ unit circle

$x_1^2 + x_2^2 = r^2 \Leftrightarrow \left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 = 1$

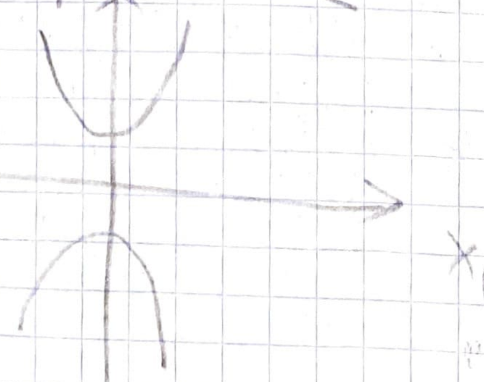
(2) $\left(\frac{x_1}{r_1}\right)^2 + \left(\frac{x_2}{r_2}\right)^2 = 1$ ellipse

(3) hyperbola

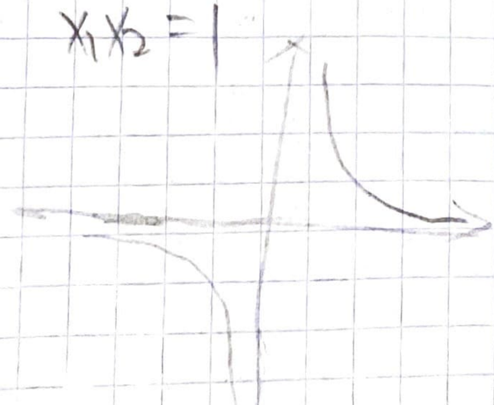
$x_1^2 - x_2^2 = 1$



$-x_1^2 + x_2^2 = 1$

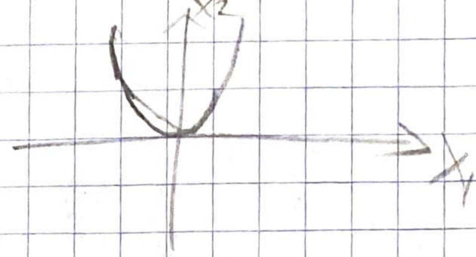


$x_1x_2 = 1$



(3) parabola

$x_1^2 = x_2$



Starting from the following 3 examples:

$x_1^2 + x_2^2 = 1$

$x_1^2 - x_2^2 = 1$

$x_2 = x_1^2$

Using linear transformation, we can obtain all possible conic curves

Linear Transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Def 1: using matrix

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} \longmapsto \begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}a_{11}x_1+a_{12}x_2\\a_{21}x_1+a_{22}x_2\end{pmatrix}$$

Def 2: $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation

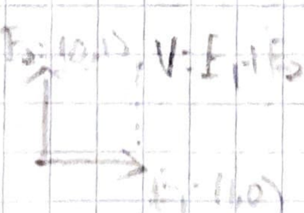
if (1) $T(c\cdot\vec{v}) = c\,T(\vec{v})$ $\forall c \in \mathbb{R}$

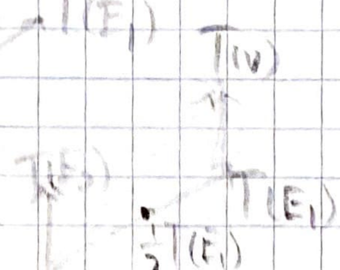
(2) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

i.e. T respects preserves the linear structure on both sides (w/o translation)

Fix a basis $\{E_1, E_2\}$ of $\mathbb{R}^2$

Prop 1: A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by how it acts on the basis vectors





$T(v) = T(E_1 + E_2) = T(E_1) + T(E_2)$

$T(w) = T(\frac{1}{2}E_1) = \frac{1}{2}T(E_1)$

Prop1 pf: Any vector $V \in \mathbb{R}^2$ can be written as

$V = a_1 E_1 + a_2 E_2$ (by def of the basis vectors)

By linearity of T, $T(V) = T(a_1 E_1 + a_2 E_2)$

$= T(a_1 E_1) + T(a_2 E_2)$

$= a_1 T(E_1) + a_2 T(E_2)$

Hence $T(V)$ is determined

- $T(E_1) = a_{11} E_1 + a_{12} E_2$ then $T(c_1 E_1 + c_2 E_2) = \_\_ E_1 + \_\_ E_2$

$T(E_2) = a_{21} E_1 + a_{22} E_2$ $= c_1 T(E_1) + c_2 T(E_2)$

$= c_1 (a_{11} E_1 + a_{12} E_2) + c_2 (a_{21} E + a_{22} E_2)$

$T\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = A \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$

$= (c_1 a_{11} + c_2 a_{21}) E_1 + (c_1 a_{12} + c_2 a_{22}) E_2$

$= (c_1, c_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$

- matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$(A \cdot B)_{ij} = \sum_k a_{ik} b_{kj}$ $(AB)_{11} = a_{11} b_{11} + a_{12} b_{21}$

ith row
jth column

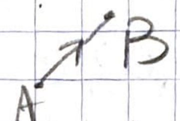
in general

$n\{\underbrace{[\ A\ ]}_{l} \cdot l\{\underbrace{[\ B\ ]}_{m} = n\{\underbrace{[\ AB\ ]}_{m}$

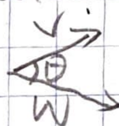
Recap:

(1). Geometric Def of a vector:

- directed segment: $\vec{AB}$



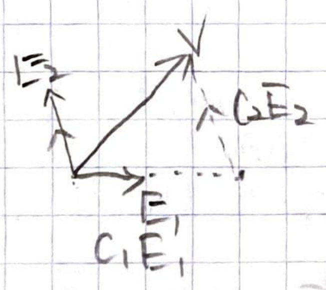
- vectors as dir. seg. up to translation
- scalar multiplication $3 \cdot \vec{V}$
  addition: put two vectors "one after another"
- inner product: $\langle \vec{V}, \vec{W} \rangle = |\vec{V}| \cdot |\vec{W}| \cdot \cos\theta$



(2) basis and coordinates:

if $\{E_1, E_2\}$ is a basis, if and only if for any $V \in \mathbb{R}^2$, there exist a unique way to write $V = C_1E_1 + C_2E_2$

The coefficients (C_1, C_2) are called coordinate of v with respect to (E_1, E_2)



$Q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$ homogeneous degree 2 polynomial

$\vec{x} = (x_1, x_2)$ in x_1, x_2 variables

Given Q quadratic form, we can try to solve $Q(\vec{x}) = 1$

$\rightsquigarrow \begin{cases} \cdot \text{ ellipse } 0 \\ \cdot \text{ hyperbola })(\\ \cdot \text{ empty curve} \end{cases}$

- Linear Transformation: $T: \mathbb{R}^2 \to \mathbb{R}^2$

Def: linear transformation is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that respect "linear structure" $\Rightarrow$ sc. multipl. addition.

Prop: $T(\vec{0}) = \vec{0}$

- $T(c_1E_1 + c_2E_2) = c_1T(E_1) + c_2T(E_2)$

hence, if we know $T(E_1)$, $T(E_2)$, we know $T(v)$ for any $v \in \mathbb{R}^2$

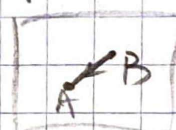
Affine Linear Transformation

$A+B = ?$ not well-defined (no origin)

$A-B = ?$ $\vec{BA}$ a directed segment $\rightsquigarrow$ a vector

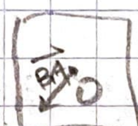
(terminology: · vector space = linear space $\xrightarrow{\text{forget the origin}}$
· affine space = affine linear space $\xrightarrow{\text{choose an origin}}$)

- $\mathbb{A}^2$ to mean 2-dim affine linear space



$\mathbb{A}^2$

$\mathbb{R}^2$ to mean linear space



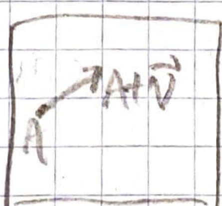
$\mathbb{R}^2$

- $-: \mathbb{A}^2 \times \mathbb{A}^2 = \mathbb{R}^2$

$(A, B) \longmapsto A - B = \vec{BA}$

- $+: \mathbb{A}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{A}^2$

$(A, \vec{v}) \longmapsto A + \vec{v}$



$\mathbb{A}^2$

notation:

- set: a collection of objects
- if A, B are sets, a map $f: A \to B$ is an assignment of elements in B to elements in A
- Given 2 sets A, B, we can define Cartesian product $A \times B = \{(a,b) \mid a \in A, b \in B\}$

$T: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$

$(x_1, x_2) \longmapsto (\tilde{x}_1, \tilde{x}_2)$ $\quad \tilde{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{10}$

$\tilde{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{20}$

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}$$

Prop: Any quadratic form $Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ can be written in a new coordinate $(\tilde{x}_1, \tilde{x}_2)$ as one of the following form

① $\tilde{x}_1^2 + \tilde{x}_2^2$ where $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

② $\tilde{x}_1^2 - \tilde{x}_2^2$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$

③ $\tilde{x}_1^2$

④ 0 ⑤ $-\tilde{x}_1^2 - \tilde{x}_2^2$

assuming: $a \neq 0$

$$
\begin{aligned}
&ax_1^2 + bx_1x_2 + cx_2^2 \\
&= a\left[x_1^2 + \frac{b}{a}x_1x_2\right] + cx_2^2 \\
&= a\left[x_1^2 + 2x_1\frac{b}{2a}x_2 + \left(\frac{b}{2a}x_2\right)^2 - \left(\frac{b}{2a}x_2\right)^2\right] + cx_2^2 \\
&= a\left[x_1 + \frac{b}{2a}x_2\right]^2 - a\cdot\frac{b^2x_2^2}{4a^2} + cx_2^2 \\
&= a\left[x_1 + \frac{b}{2a}x_2\right]^2 + \left(c - \frac{b^2}{4a}\right)\cdot x_2^2
\end{aligned}
$$

① if $a > 0$, $c > \frac{b^2}{4a}$, then define $\tilde{x}_1 = \sqrt{a}\left(x_1 + \frac{b}{2a}x_2\right)$

$\tilde{x}_2 = \sqrt{c - \frac{b^2}{4a}}\; x_2$

$Q = \tilde{x}_1^2 + \tilde{x}_2^2$

② if $a < 0$, $c < \frac{b^2}{4a}$, $\tilde{x}_1 = \sqrt{|a|}\left(x_1 + \frac{b}{2a}x_2\right)$

$\tilde{x}_2 = \sqrt{\left|c - \frac{b^2}{4a}\right|}\; x_2$

Prop 2: $Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f$ can be writen in a new affine linear coordinate $(\tilde{x}_1, \tilde{x}_2)$ as one of the following form

① $\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{c}$ ⑤ $-\tilde{x}_1^2 - \tilde{x}_2^2 + \tilde{c}$

② $\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{c}$ where $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}$

③ $\tilde{x}_1^2 + \tilde{c}$

④ $\tilde{c}$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix}$

$$Q = x^2 + y^2 + 2ax + 2by + c$$
$$= (x+a)^2 - a^2 + (\cancel{x}y+b)^2 - b^2 + c$$
$$= (x+a)^2 + (\cancel{x}y+b)^2 + (c - a^2 - b^2)$$
$$= \tilde{x}^2 + \tilde{y}^2 + \tilde{c}$$

Review: Linear Transformation and Matrices

- $T: R^2 \longrightarrow R^2$ $\quad T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

$$T(c\cdot\vec{v}) = cT(\vec{v})$$

. Let $\{\vec{E}_1, \vec{E}_2\}$ be a basis of $\mathbb{R}^2$

and suppose $(T\vec{E}_1, T\vec{E}_2) = (\vec{E}_1, \vec{E}_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (\vec{E}_1 a_{11} + \vec{E}_2 a_{21},\ \vec{E}_1 a_{12} + \vec{E}_2 a_{22})$

Let $\vec{v} \in \mathbb{R}$ be any vector

Then we can write

$$\vec{v} = c_1\vec{E}_1 + c_2\vec{E}_2 \text{ for some } c_1, c_2 \in \mathbb{R}$$

$$= (\vec{E}_1, \vec{E}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$T\vec{v} = T(\vec{E}_1, \vec{E}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (T\vec{E}_1, T\vec{E}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (\vec{E}_1, \vec{E}_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= (\vec{E}_1, \vec{E}_2)\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}$$

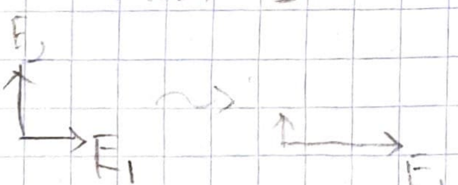
$$\tilde{c}_1 = a_{11}c_1 + a_{12}c_2$$

$$\tilde{c}_2 = a_{21}c_1 + a_{22}c_2$$

Ex: (1) stretching

$T(\vec{E}_1) = 2\vec{E}_1$

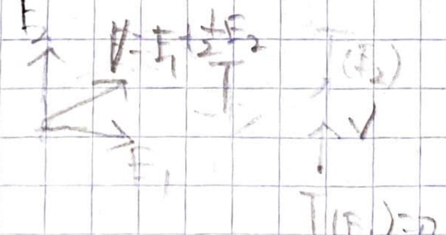
$T(\vec{E}_2) = \frac{1}{3}\vec{E}_2$



(2) projection:

$T(E_1) = 0$

$T(E_2) = E_2$



$T(\vec{v})$

$= T(\vec{E}_1 + \frac{1}{2}\vec{E}_2)$

$= \frac{1}{2}\vec{E}_2$

image of $\mathbb{R}^2$ under T is the subspace $\mathbb{R}\cdot\vec{E}_2$

(3) another projection:

$T(E_1) = \frac{1}{2}E_2$

$T(E_2) = E_2$

Operations on Linear Transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$

(1) addition:
Scalar multiplication

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$
$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$

$(c_1T_1 + c_2T_2)(\vec{v})$
$= c_1T_1(\vec{v}) + c_2T_2(\vec{v})$

$c_1T_1 + c_2T_2 : \mathbb{R}^2 \to \mathbb{R}^2$
is still linear transformation

if T_1 is represented by matrix $[T_1] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$[T_2] = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ then $c_1T_1 + c_2T_2$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^2 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

$\vec{v} \longmapsto T_1(\vec{v}) \longrightarrow T_2(T_1(\vec{v}))$

Ex $[T_1] = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

$[T_2 \circ T_1] = [T_2] \cdot [T_1]$

$[T_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$[T_2] \cdot [T_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 2 & 0 \end{pmatrix}$

Remark: (1) $[T_1] \circ [T_2] \neq [T_2][T_1]$

(2) $([T_1] \cdot [T_2])[T_3] = [T_1] \cdot ([T_2][T_3])$

(3) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix $[I] \cdot [T] = [T]$

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible, if there is an $S: \mathbb{R}^2 \to \mathbb{R}^2$

$S \circ T = Id$

Lemma:

$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $[T]$ is inverteble, if and only is $\det([T]) \neq 0$

$ad - bc$

then $[T] \cdot [T]^{-1} = [Id]$

$[T]^{-1} \cdot [T] = [Id]$

$$[T]^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$T(E_2) = \begin{pmatrix} b \\ d \end{pmatrix}$

$T(E_1) = \begin{pmatrix} a \\ c \end{pmatrix}$ $\quad |\text{Area}(\vec{E_1}, \vec{E_2})| = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\vec{v}, \vec{w}$ are linear independent

if $c_1\vec{v} + c_2\vec{w} = 0 \overset{\text{if and only if}}{\Longleftrightarrow} c_1 = 0$ and $c_2 = 0$

Recall: • Let $\mathbb{R}^2$ be equiped with the standard inner product

$\langle \vec{u}, \vec{v} \rangle = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$

• A basis $\{\vec{E_1}, \vec{E_2}\}$ is orthonomal, if $|\vec{E_i}| = 1$

$\langle \vec{E_1}, \vec{E_2} \rangle = 0$

if $\vec{v}$ has coord $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\vec{v} = \vec{E_1} v_1 + \vec{E_2} v_2$

and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

then $\langle \vec{v}, \vec{w} \rangle = v_1 w_1 + v_2 w_2$

Def: A linear trans. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal trans.

if for any $\vec{v}, \vec{w} \in \mathbb{R}$ $\langle T\vec{v}, T\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$

then $\langle \vec{v} \cdot \vec{w} \rangle = v_1 w_1 + v_2 w_2 = (w_1, w_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (v_1, v_2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$= \vec{w}^t \cdot \vec{v} = \vec{v}^t \cdot \vec{w}$

Prop: Let $\{\vec{e_1}, \vec{e_2}\}$ be one ONB of $\mathbb{R}^2$

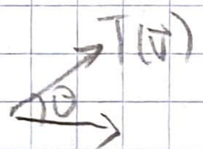
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear trans.

and $[T]$ the matrix form with respect to $\{\vec{e}_1, \vec{e}_2\}$

Then T is an orthogonal trans.

iff $[T^{-1}]^{-1} = [T]^t$

Ex. (1) $[T] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$(R_\theta)^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$[R(\theta)]^t [R_\theta] = \begin{pmatrix} c^2+s^2 & 0 \\ 0 & c^2+s^2 \end{pmatrix}$

Recall: $\cdot$ vector space $\mathbb{R}^2$
$\cdot$ inner product $\langle\ ,\ \rangle$ $\begin{cases} \text{linear in each slot} \\ \langle u,u\rangle \geq 0 \\ \text{symmetric } \langle u,w\rangle = \langle w,u\rangle \end{cases}$

$\cdot$ $T: \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal transformation of $\mathbb{R}^2$ $T \in O(2)$

if $\forall\, u,v \in \mathbb{R}^2$ $\langle u,v\rangle = \langle Tu, Tv\rangle$

Prop: if $T: \mathbb{R}^2 \to \mathbb{R}^2$ preserves Length, then $\langle u,v\rangle = \langle Tu, Tv\rangle$

Proof: By polarization, we have

$$\langle u+v, u+v\rangle = \langle T(u+v), T(u+v)\rangle$$

$$\langle u,u\rangle + \langle v,v\rangle + 2\langle u,v\rangle = \langle Tu, Tu\rangle + \langle Tv, Tv\rangle + 2\langle Tu, Tv\rangle$$

$$\Rightarrow \langle u,v\rangle = \langle Tu, Tv\rangle$$

Given an orthonormal basis $(\vec{e}_1, \vec{e}_2)$, then an ortho $T \in O(2)$ can be presented as $[T] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
so that given a vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$T\vec{v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Prop $[T]^{-1} = [T]^t$ $(T \in O(2))$ i.e. $[T]^t[T] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Proof: For any 2 vectors $\vec{v}, \vec{w} \in \mathbb{R}^2$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\langle \vec{v}, \vec{w}\rangle = (v_1, v_2)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (v_1, v_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\langle T\vec{v}, T\vec{w}\rangle = \left([T]\vec{v}\right)^t \cdot \left([T]\vec{w}\right)$$

$$= \vec{v}^t \cdot [T]^t [T] \cdot \vec{w}$$

Lemma: Let M be a 2×2 matrix
if for any $\vec{v}, \vec{w} \in \mathbb{R}^2$
We have $\vec{v}^t \cdot M \cdot \vec{w} = 0$
then $M = 0$

Pf: $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ if M is not zero
then at least one entry of M is non zero
say $m_{11} \neq 0$
then we can let $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$0 = \vec{v}^t M \cdot \vec{w} = (1,0)\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ Contradiction
$$= (1,0)\begin{pmatrix} m_{11} \\ m_{22} \end{pmatrix} = m_{11}$$

In fact
$m_{ij} = e_i^t \cdot M e_j$ Thus $m_{ij} = 0 \ \forall i, j$

Since $0 = \langle \vec{v}, \vec{w} \rangle - \langle T\vec{v}, T\vec{w} \rangle$
$= \vec{v}^t \cdot I_2 \vec{w} - \vec{v}^t [T]^t [T] \cdot \vec{w}$
$= \vec{v}^t \cdot (I_2 - [T]^t [T]) \vec{w}$

By the previous lemma, we have $I_2 - [T]^t [T] = 0$

$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\det R(\theta) = \cos^2\theta - (\sin\theta)(-\sin\theta)$
$= \cos^2\theta + \sin^2\theta = 1$

Rotation matrix

$R(\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

A B (2×2)

$R(\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$\det(A\cdot B) = (\det A)(\det B)$

$\det(A) = \det$

Inverses

A is invertible iff $\exists$ B such that $AB = I$. we write $B = A^{-1}$

1) If $AB = I \Rightarrow BA = I$

2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$ad - bc \neq 0$

$x_1 = ax_1' + bx_2'$
$x_2 = cx_1' + dx_2'$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$

$A^{-1}\vec{x} = \vec{x}'$

Similarity Transformations

$\vec{y} = T\vec{x}$ $\quad$ $T\vec{x} = A\vec{y}'$

$\vec{x} = A\vec{x}'$ $\quad$ $A^{-1}T\vec{x} = \vec{y}'$

$\vec{y} = A\vec{y}'$ $\quad$ $A^{-1}TA\vec{x}' = \vec{y}'$

$A^{-1}TA$ is the matrix of the L.T. that sends $\vec{x}' \mapsto \vec{y}'$

(related)

$A^{-1}TA$ and T are similar by a similarity transformation

Prove that z^{-1}, $\bar{z}$ are proportional -

$z = a + bi$ $\quad$ $\bar{z} = a - bi$

$z^{-1} = \frac{1}{a+bi}$

$\frac{\bar{z}}{z^{-1}} = (a-bi)(a+bi)$

$= a^2 + b^2$

$z^{-1} = \left(\frac{1}{a^2+b^2}\right)\bar{z}$

Compute $\left(\frac{\sqrt{3}+i}{2}\right)^{100} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{100} = \left(e^{i\frac{\pi}{6}}\right)^{100} = e^{\frac{50i\pi}{3}}$

$= \cos\frac{50\pi}{3} + i\sin\frac{50\pi}{3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= \frac{-1+i\sqrt{3}}{2}$

Express $\cos(\theta_1+\theta_2)$ in terms of $\cos(\theta_i)$ $\sin(\theta_i)$ $i = 1, 2$

$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$e^{i(\theta_1+\theta_2)} = (e^{i\theta_1})(e^{i\theta_2})$

$\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$
$+ i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1$

$z^3+8=0$

$(z+2)(z^2-2z+4)=0$

$z=-2\ ,\ z^2-2z+4=0$

$z=\frac{2\pm\sqrt{-12}}{2}=1\pm\sqrt{-3}$

$=1\pm i\sqrt{3}$

$z^3+i=0$

$(z+\sqrt[3]{i})(z^2-\sqrt[3]{i}z-1)=0$

$z=-\sqrt[3]{i}\ ,\quad z=\frac{\sqrt[3]{i}\pm\sqrt{3}}{2}$

$z^4+4z^2+4=0$

$(z^2+2)^2=0$

$z^2=-2$

$z=\pm i\sqrt{2}$

$z^4-2z^2+4=0$

$z^4+1=0$

$z^4-(i)^2=0$

$(z^2-i)(z^2+i)=0$

$z^2=i \quad z^2=-i$

$z=\pm\sqrt{i} \quad z=\pm\sqrt{-i}$

- Complex Numbers
  - i or $\sqrt{-1}$ as the one of the solution to $x^2+1=0$

- a complex number: $z = a + bi$, $a, b \in \mathbb{R}$ — a: the real part, b: the imaginary part — $\sqrt{-1}$, $-\sqrt{-1}$

  $Re(z) = a$; $Im(z) = b$

  $(a+bi)+(c+di) = (a+c)+(b+d)i$

  $(a+bi)(c+di) = ac + bci + adi - bd = (ac-bd) + (bc+ad)i$

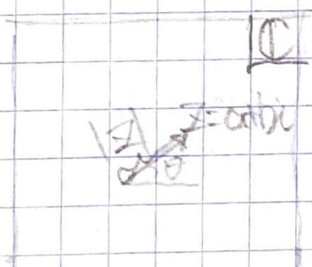
- complex conjugation:

  $z = a+bi$ $\quad \bar{z} = a+b(-i)$ $\quad$ change i to $(-i)$

if we are given a degree 2 polynomial with real coeff

$$f(x) = x^2 + c_1 x + c_0 \qquad c_1, c_0 \in \mathbb{R}$$

Then the equation $f(x)=0$ sometimes has 2 complex solution — they are in complex conjugate



$|z| = \sqrt{a^2+b^2}$

$\theta = arg(z)$ ← argument (we identify θ with $\theta + 2\pi$)

polar form

$$z = r \cdot e^{i\theta} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$= r\cos\theta + ir\sin\theta$$

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1+\theta_2)} \qquad \bar{z} = re^{-i\theta} \qquad i = e^{\frac{\pi}{2}i}$$

Fundamental theorem of algebra:

Let $f(z) = z^n + c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \dots + c_0$

be a degree n polynomial where $c_i \in \mathbb{C}$

Then $f(z)$ has n solutions (counted with multiplicities)

$f(z) = (z-z_1)^{m_1} \cdots (z-z_r)^{m_r}$ $\quad m_i \geq 1$ integers

z_i are distinct roots $\quad m_1 + \dots + m_r = n$

Four Theorems of Linear Algebra

- Rank Thm
- Inertia Thm
- Orthogonal Diag. Thm
- Jordan Normal Thm

Math: Classification Problem

- Transformation (Group)
- We want to define equivalence up to transformation

$$G \curvearrowright X$$

$$a: G \times X \longrightarrow X$$

Two elements $x, y \in X$ are equivalent, if there exists $g \in G$, such that $g \cdot x = y$. We denote $x \sim y$.

$\Leftrightarrow x = g^{-1} \cdot y$

For a system of linear equations:

$$y_1 = a_{11}x_1 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + \quad + a_{2n}x_n$$

$$\vdots$$

$$y_m = a_{m1}x_1 + \cdots + a_{mn}x_n$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\vec{y} = A \cdot \vec{x} \qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

① Describe the equivalence class

$$X/\sim$$

an equivalence class can be described by a model element a.k.a representative

Ex: {conic curves}

= {ellipses} ∪ {parabolas} ∪ {hyperbolas}

unit circle  $y^2 = x$  $x^2 - y^2 = 1$

Rank Thm:

$\vec{y} \in M_{m\times 1}$

Any linear equation of the form

$\vec{x} \in M_{n\times 1}$

$\vec{y} = A\vec{x}$, where $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Can be simplified by a "change of variables"

$\vec{y} = B\vec{Y}$, $\vec{x} = C\vec{X}$ so that

$\vec{Y} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \\ 0 & & 1 & & \\ \vdots & & & & \\ 0 & 0 & \cdots & & 0 \end{pmatrix} \vec{X}$

B is $m\times m$ invertible

C is $n\times n$ invertible

$A \in M_{m\times n}$

$Y_1 = X_1$

$Y_2 = X_2$

$\vdots$

$Y_r = X_r$

$Y_{r+1} = 0$

$\vdots$

$Y_m = 0$

Ex: $T: \underset{\vec{x}}{\mathbb{R}} \to \underset{\vec{y}}{\mathbb{R}^2}$

$t \longmapsto (2t, 3t)$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} (x_1)$

$\begin{cases} Y_1 = 2y_1 \\ Y_2 = 3y_1 - 2y_2 \end{cases}$ $\vec{X} = \vec{x}$

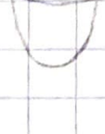
$\begin{cases} Y_1 = x_1 \\ Y_2 = 0 \end{cases}$

$T: \underset{\vec{x}}{\mathbb{R}^2} \to \underset{\vec{y}}{\mathbb{R}^2}$

$\vec{y} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x}$

Graph of functions

$f_1(x,y) = x^2 + y^2$



$f_2(x,y) = x^2 - y^2$

$f_3(x,y) = -x^2 - y^2$

Given a quadratic form

$Q(\vec{x}) = \vec{x}^t \cdot M \cdot \vec{x}$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

symmetric $a_{ij} = a_{ji}$

Up to linear transformation

$\vec{x} = A \cdot \vec{X} \qquad A \in GL(\mathbb{R}^n)$

$\uparrow$ $n \times n$ invertible matrices

$Q(\vec{x}) = \vec{x}^t \cdot M \cdot \vec{x} = (A \cdot X)^t \cdot M \cdot (A \cdot X)$

$= \vec{X}^t \cdot (A^t \cdot M \cdot A) \cdot \vec{X}$

$= \vec{X}^t \cdot \Lambda \cdot \vec{X}$

$$\Lambda = \begin{pmatrix} \overbrace{1 \;\ddots}^{p} & \overbrace{-1}^{q} & \overbrace{0 \;\ddots}^{n-p-q} & 0 \\ & \ddots & -1 & \\ 0 & & & 0 \ddots 0 \end{pmatrix}$$

input $\vec{y} = A \vec{x}$

want to find $\vec{y} = B \vec{Y}$

$\vec{x} = C \vec{X}$

$B \vec{Y} = A \cdot C \cdot \vec{X}$

$\vec{Y} = \underbrace{B^{-1} \cdot A \cdot C}_{\Lambda} \cdot \vec{X}$

Orthogonal Diagonalization Thm

Given

- $Q(\vec{x}) = \vec{x}^t \cdot M \cdot \vec{x}$

want to find : $A \in \overset{\text{Orthogonal group}}{O}(\mathbb{R}^n)$

$= \{ M \in GL[n] \mid M^t M = I \}$

$= \{ M : \mathbb{R}^n \to \mathbb{R}^n \mid \langle Mu, Mv \rangle = \langle u, v \rangle \ \forall u, v \in \mathbb{R}^n \}$

s.t. if we set $\vec{x} = A \cdot \vec{X}$

then $\vec{x}^t \cdot M \cdot \vec{x} = \vec{X}^t \underbrace{(A^t \cdot M \cdot A)}_{\Lambda} \vec{X}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Jordan Normal Form

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\vec{y} = A \cdot \vec{x}$$

Want to find change of basis of $\mathbb{R}^n$

$$\vec{x} = B\vec{X} \qquad B \in Gl(n)$$

$\vec{y} = B\vec{Y}$ s.t. the equation in $\vec{X}$, $\vec{Y}$ simplifies

$$B\vec{Y} = A \cdot B \cdot \vec{X}$$

$$Y = \underbrace{B^{-1} \cdot A \cdot B}_{\Lambda} \vec{X}$$

Λ consists of block diagonal matrices

$$\begin{array}{c} T_1 \\ T_2 \\ \\ \end{array} \begin{pmatrix} \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} & & 0 \\ 0 & \begin{pmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} & 0 \\ 0 & & \ddots \end{pmatrix}$$

each block is

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 0 \end{pmatrix}$$

Jordan Block

Notation: • $A = [a_{ij}]$ i: row index j: column index

• transpose A: $n \times m$ matrix $A = [a_{ij}]$

$$(A^t)_{ij} = A_{ji}$$

• matrix multiplication:

A: $n \times m$ B: $m \times l$ $A \cdot B$: $n \times l$

$$(A \cdot B)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

Definition of Determinant:

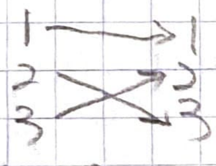
Let A be a $n \times n$ matrix

• $\det A$ is a number defined by $\det(A) = \sum_{\substack{\sigma \\ \text{permutation} \\ \text{of } (1,2,\dots,n)}} \varepsilon(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

a permutation of $\{1, 2, \dots, n\}$ "S_n"

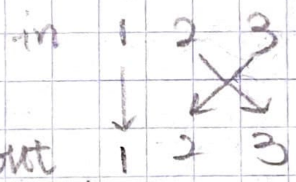
is a bijection

$\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$

Ex:  number of permutations of size n is $n!$

• length of a permutation: $L(\sigma) \geq 0$ integer

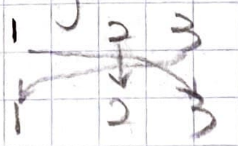
• geometrical definition

$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ 

?

$L(\sigma) = \#$ of pairs (i, j) that swap ordering $i < j$ and $\sigma(i) > \sigma(j)$

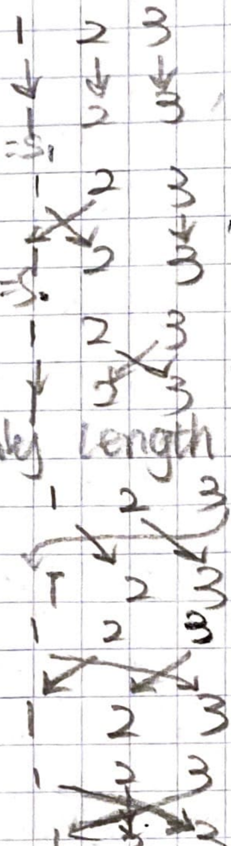
= number of crossing

Ex: $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ 

Rank: there is exactly 1 element $\sigma \in S_n$ that $L(\sigma) = \frac{n(n-1)}{2}$

$$\varepsilon(\sigma) = (-1)^{L(\sigma)} = \begin{cases} +1 & L(\sigma) \text{ even} \\ -1 & L(\sigma) \text{ odd} \end{cases}$$

$n=3$:



- $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $L(\sigma)=0$, $\varepsilon(\sigma)=1$

$S_{12}=S_1$

- $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $L(\sigma)=1$, $\varepsilon(\sigma)=-1$

$S_{23}=S_2$

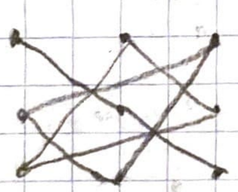
- $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $L(\sigma)=1$, $\varepsilon(\sigma)=-1$

These 2 are the only length 1 permutation

- $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $L(\sigma)=2$ +
- $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $L(\sigma)=2$ +
- $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $L(\sigma)=3$ − ?

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma\in S_3} \varepsilon(\sigma)\cdot a_{1\sigma(1)}\cdot a_{2\sigma(2)}\cdot a_{3\sigma(3)}$$
$$= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

(+):  (−): 

S_n: permutation group $[n]=\{1,2,\cdots,n\}$

given $\sigma_1 : [n] \longrightarrow [n]$

$\sigma_2 : [n] \longrightarrow [n]$

I can compose then

$\sigma_2\circ\sigma_1 : [n]\longrightarrow[n]$

$i \longmapsto \sigma(\sigma_1(i))$

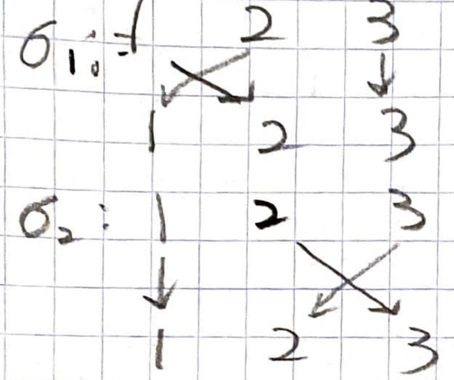
map of sets

$f: A\to B$

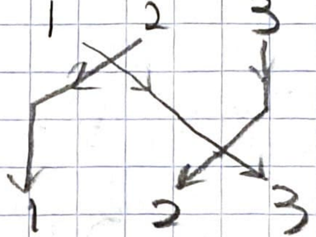
$g: B\to C$

$g\circ f: A\to C$

$a \longmapsto g(f(a))$



$$(\sigma_2 \cdot \sigma_1) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$



Lemma: Let $\sigma \in S_n$

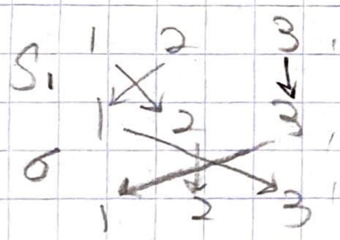
Let S_i be a simple transposition swapping $i, i+1$

then $L(\sigma \cdot S_i) = \begin{cases} L(\sigma)+1 & \text{if } \sigma(i) < \sigma(i+1) \\ L(\sigma)-1 & \text{if } \sigma(i) > \sigma(i+1) \end{cases}$?

Proof: Consider any pair (j, k) with $j<k$

- if $(j,k) \neq (i, i+1)$, then $\sigma(j) < \sigma(k) \iff \sigma S_i(k)$
- if $(j,k) = (i, i+1)$, then $\sigma(i) < \sigma(i+1) \iff \sigma S_i(i) > \sigma S_i(i+1)$

$S_1 \cdot S_1 = id$



Prop: if $\sigma \in S_n$ has length $L(\sigma)$ then σ can be written as

$\sigma = S_{i_{L(\sigma)}} \cdots S_{i_2} S_{i_1}$

where $i_1, \dots, i_{L(\sigma)} \in \{1, \dots, n-1\}$?

? Proof: We prove by induction. The statement is true for $L(\sigma) = 1, 0$. Suppose the statement is true for all those σ with $L(\sigma) \le k-1$

And suppose $L(\sigma) = k$

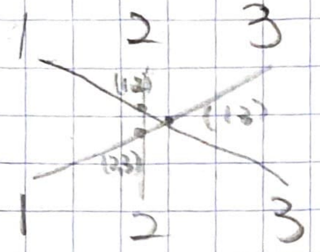
- Find a S_i such that $L(\sigma \cdot S_i) = L(\sigma) - 1 = k-1$

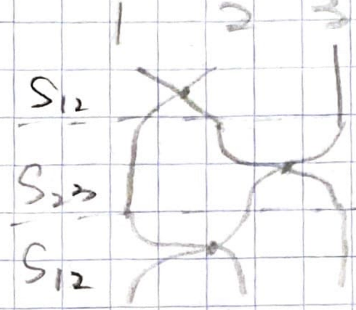
then by induction hypothesis

$(\sigma S_i) = S_{i_{k-1}} \quad S_{i_1}$

$\sigma = (\sigma S_i) \cdot S_i = S_{i_{k-1}} \quad S_{i_1} S_i$

- if no such S_i exists, then $\sigma(i) < \sigma(i+1)$ then $\sigma = id$, which we already cover





$\sigma = S_{12} S_{23} S_{12}$
$= S_{23} S_{12} S_{23}$

A: $n \times n$ matrix with A_{ij} the entry on ith row jth col

- $\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot A_{1\sigma(1)} \cdot A_{2\sigma(2)} \cdots A_{n\sigma(n)}$

Where S_n is the group of permutation of n elements

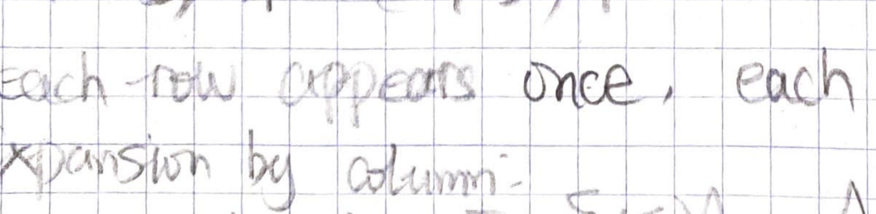
$\varepsilon(\sigma) = (-1)^{l(\sigma)}$

$l(\sigma)$ = minimum number of elementary swapping to obtain σ

Ex: $n = 3$

$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, (2\ 3\ 1)\ (3\ 1\ 2)\ (3\ 2\ 1) \right\}$

- Alternative definition



$(1\ 2\ 3)\ +1$ $(2\ 1\ 3)\ -1$ $(2\ 3\ 1)\ +1$

Each row appears once, each column appears once

Expansion by column:

Claim: $\det(A) = \sum_{\sigma} \varepsilon(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n}$ $A_{\sigma^{-1}(3)3}$

Pf: $n=3$ By def. $\det(A) = \sum_{\sigma} \varepsilon(\sigma) A_{\sigma^{-1}(1)1} A_{\sigma^{-1}(2)2} \cdots$

Rename σ^{-1} to σ

$= \sum \varepsilon(\sigma^{-1}) A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3}$

$= \sum \varepsilon(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3}$

Properties of det:

① $\det(A^t) = \det(A)$

② Interchange any 2 columns would change the sign

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$$

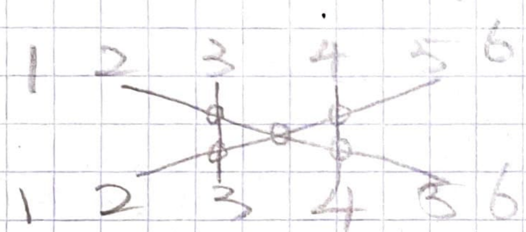
$\det([\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n])$

swapped $\vec{a}_i$ and $\vec{a}_j$

$= (-1) \det([\vec{a}_1, \cdots, \vec{a}_{i-1}, \vec{a}_j, \vec{a}_{i+1}, \cdots, \vec{a}_{j-1}, \vec{a}_i, \vec{a}_{j+1}, \cdots, \vec{a}_n]$

Lemma: if σ is a transposition (ij), then $\varepsilon(\sigma) = -1$

Pf:



$\sigma = (ij)$

$l(\sigma)$ is odd

$\varepsilon(\sigma) = (-1)^{odd} = -1$

③ $\det(\vec{a}_1, \vec{a}_2, \cdots \lambda\vec{a}_i, \vec{a}_{i+1}, \cdots, \vec{a}_n)$

$= \lambda \cdot \det(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_{i-1}, \vec{a}_i, a_{i+1}, \cdots, \vec{a}_n)$

for any $\lambda \in \mathbb{R}$

④ $\det(\vec{a}_1, \vec{a}_2 + \vec{a}_2', \vec{a}_3)$

$= \det(\vec{a}_1, \vec{a}_2, \vec{a}_3) + \det(\vec{a}_1, \vec{a}_2', \vec{a}_3)$

③④ $\Leftrightarrow \det(\_, \_, \cdots, \_)$ is a multilinear function $\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ factor}} \to \mathbb{R}$

⑤ $\det I = 1$

⑥ $\det(\vec{a}_1, \cdots \vec{a}_n) = 0$ if one of the $\vec{a}_i = 0$

⑦ $\det(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n) = 0$, if there are i, j such that $\vec{a}_i = \vec{a}_j$

$\det(A) = \sum (-1)^{\varepsilon(\sigma)} \cdot \underbrace{a_{\sigma(1)1} \cdots a_{\sigma(n)n}}_{= S(\sigma)}$

if $a_{\cdot i} = a_{\cdot j}$, then

$S(\sigma) = S(\sigma(ij))$

but $(-1)^{\sigma} = -(-1)^{\sigma(ij)}$

⑧ Fix i, j, $\lambda \in \mathbb{R}$

$\det(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$

$= \det(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_i + \lambda \vec{a}_j, \ldots, \vec{a}_n)$

RHS $= \det(a_1, \ldots a_n) + \lambda \det(a_1, \ldots, \vec{a}_j, \ldots \vec{a}_j, \ldots \vec{a}_n)$

$= \det(a_1 \ldots a_n) + 0$

⑨ $\det \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = \lambda_1 \lambda_2 \cdots \lambda_n$

LHS $= \lambda_1 \cdots \lambda_n \det \begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \lambda_1 \cdots \lambda_n$

$$M = \begin{array}{c c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$$

- if $C = 0$, then $\det M = (\det A) \cdot (\det D)$

Prop: if A, B are $n \times n$ matrices

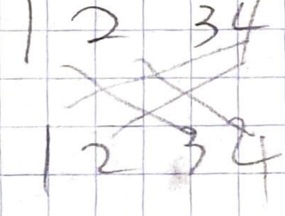
then $\det(A \cdot B) = (\det A) \cdot (\det B)$

Pf: $\det(A)\cdot\det(B) = \det \underbrace{\begin{pmatrix} A & 0 \\ -I & B \end{pmatrix}}_{2n} \Big\} 2n$

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{pmatrix} \xrightarrow{\times b_{11},\ \times b_{21}} \begin{pmatrix} a_{11} & a_{12} & b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, & 0 \\ & & & 0 \\ -1 & 0 & 0 & b_{12} \\ 0 & -1 & 0 & b_{22} \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} A & A\cdot B \\ -I & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$



$4 = 2n$

crossing $= n^2$?

$= (-1)^{n^2} \begin{pmatrix} AB & A \\ 0 & -I \end{pmatrix}$

$= (-1)^n \det(AB)\det(-I_n) = (-1)^n(-1)^n \det(AB)\det(I)$

$= \det(AB)$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] \arrow[rr, bend right, "A\cdot B"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

Cofactor

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A) = ad - bc$

if $\det A \neq 0$

then $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^{-1} \cdot A = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{array}{c c} & j \\ i & \begin{pmatrix} & \vdots & \\ \cdots & \cdot & \cdots \\ & \vdots & \end{pmatrix} \end{array}$$

$C_{ij} = (-1)^{i+j} \det(A$ remove ith row jth column$)$

Cofactor Thm:

A: $n\times n$ matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & & A_{nn} \end{bmatrix}$$

Given $1\le i \le j \le n$, we form a new (sub) matrix by deleting the ith row and jth column matrix.

Let $C_{ij} = (-1)^{i+j} \cdot \det(\text{this } (n-1)\times(n-1) \text{ matrix})$
↑ cofactor

Thm: $\det(A) \overset{!}{=} A_{11}C_{11} + A_{12}\cdot C_{12} + \cdots + A_{1n}C_{1n}$ (first row)

$= A_{11}C_{11} + A_{21}C_{21} + \cdots + A_{n1}C_{n1}$ (first column)

• In general, for $1\le i_1, i_2 \le n$, we have $\sum_{j=1}^{n} A_{i_1 j} C_{i_2 j} = \begin{cases} \det(A) & i_1 = i_2 \\ 0 & i_1 \ne i_2 \end{cases}$

Similarly $\sum_{j=1}^{n} A_{j i_1} C_{j i_2} = (\delta_{i_1 i_2}) \det(A)$

$= \det(A)\cdot \delta_{i_1 i_2}$

$\begin{cases} 1 & i_1 = i_2 \\ 0 & i_1 \ne i_2 \end{cases}$

$\det(A) = \sum_{\sigma \in S_n} \cdot \varepsilon(\sigma) \cdot A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$

$= \underbrace{A_{11} \cdot \det\begin{pmatrix}\text{matrix } A \\ \text{delete 1st row} \\ \text{1st col}\end{pmatrix}}_{\text{All the terms with } \sigma(1)=1} - \underbrace{A_{12} \cdot \det\begin{pmatrix} A \text{ remaining} \\ \text{1st row} \\ \text{2nd col}\end{pmatrix}}_{\text{All the terms with } \sigma(1)=2} + A_{13} \det\begin{pmatrix} A - \text{1st row} \\ - \text{3rd col}\end{pmatrix} \cdots$

$= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + \cdots + A_{1n}C_{1n}$

If $i_1 = i_2 = i \ne 1$, $A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} = \det A$

build a new matrix, $\tilde{A}$ that moves the i-th row to the top ...

$\det(\tilde{A}) = (-1)^{i-1}\det(A)$ $\quad \tilde{C}_{ij} = (-1)^{1+j} \det\begin{pmatrix}\tilde{A} \text{ deleting} \\ \text{1st row} \\ j\text{th col}\end{pmatrix}$

$(\tilde{A})_{1j} = A_{ij}$

$\overset{?}{=} (-1)^{1+j} \det\begin{pmatrix} A \text{ delete} \\ i\text{-th row} \\ j\text{-th col}\end{pmatrix}$

$= \underbrace{(-1)^{1+j}(-1)^{i+j}}_{(-1)^{i+1}} \underbrace{(-1)^{i+j} \det(\cdots)}_{C_{ij}} = (-1)^{i+1} C_{ij}$

$\det \tilde{A} = \sum_j \tilde{A}_{1j} \tilde{C}_{1j}$

$(-1)^{i-1} \det(A) = \sum_j A_{ij} C_{ij} (-1)^{i+1}$

$\det(A) = \sum_j A_{ij} C_{ij}$

If $i_1 \neq i_2$, say $i_1 = 1$, $i_2 = 2$, we want to show

$\sum_{j=1}^{n} A_{1j} \cdot C_{2j} = 0$ $\qquad \tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \end{bmatrix}$ $\tilde{A}$ copies the i-th row of A for $i \neq 2$, $\tilde{A}_{ij} = A_{ij}$

$0 = \det \tilde{A} = \tilde{A}_{21} \tilde{C}_{21} + \tilde{A}_{22} \tilde{C}_{22} + \cdots + \tilde{A}_{2n} C_{2n}$

$= A_{11} C_{21} + A_{12} C_{22} + \cdots + A_{1n} C_{2n}$

Corollary: if A is an $n \times n$ matrix with $\det A \neq 0$, then

$A^{-1} = \frac{1}{\det A} \cdot C^t$ ((C is the matrix of all cofactors C_{ij})

Pf: $\left[\left(\frac{1}{\det A} C^t \right) \cdot A \right]_{ij}$

$= \frac{1}{\det A} \sum_{k=1}^{n} (C^t)_{ik} A_{kj} = \frac{1}{\det A} \sum_{k=1}^{n} C_{ki} A_{kj} = \frac{1}{\det A} \cdot \det A \cdot \delta_{ij} = \delta_{ij}$

$\left[A \cdot \left[\frac{1}{\det} C^t \right] \right]_{ij} = \delta_{ij}$ by similar argument

$i = j$ only when the entry is on the diagonal

Cramer's Rule:

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a column of unknown variables,

let $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ be a given vector

Let A be a $n \times n$ matrix (invertible)

Consider equation $A\vec{x} = \vec{a}$

then $\vec{x} = A^{-1} \cdot \vec{a}$

thus

$$x_1 = \frac{\det\begin{pmatrix} a_1 & A_{12} & \cdot & A_{1n} \\ \vdots & \vdots & & \\ a_n & A_{n2} & & A_{nn}\end{pmatrix}}{\det A} = \frac{\det(\vec{a}, A_{\cdot 2}, A_{\cdot 3}, \cdots, A_{\cdot n})}{\det A}$$

$$x_i = \frac{\det(A_{\cdot 1}, \cdots, \underset{\text{ith slot}}{\vec{a}}, \cdots A_{\cdot n})}{\det A}$$

$$x_i = (A^{-1}\cdot\vec{a})_i \qquad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$= \sum_{k=1}^{n} (A^{-1})_{ik} a_k$$

$$= \sum_{k=1}^{n} \left(\frac{1}{\det A} C^t\right)_{ik} a_k$$

$a_k \Rightarrow A_{ki}$

$$= \frac{1}{\det A}\left[\sum_{k=1}^{n} C_{ki} a_k\right] = \frac{1}{\det A} \det\begin{pmatrix} A, \text{ replacing} \\ \text{the ith column} \\ \text{by } a \end{pmatrix}$$

$$A = \begin{array}{c|cc} & n_1 & n_2 \\ \hline n_1 & A_{11} & A_{12} \\ n_2 & A_{21} & A_{22} \end{array}, \qquad B = \begin{array}{cc} n_1 & n_2 \\ \hline B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}$$

One can treat the blocks as if they're numbers

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ & \end{bmatrix}$$

$$M = \begin{array}{c|cc} & n_1 & n_2 \\ \hline n_1 & A & B \\ n_2 & C & D \end{array}$$

$$\det(M) = \det(D)\cdot\det(\underbrace{A - B\cdot D^{-1}\cdot C}_{\text{size } n_1 \times n_1})$$

$$\left[M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \det M = ad - bc \underset{\substack{\uparrow \\ \text{needs } d \neq 0}}{=} d\left(a - \frac{bc}{d}\right)\right.$$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \det\left(\begin{array}{c|c} I & 0 \\ \hline -D^{-1}C & I \end{array}\right)$$

$$= \det\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} A - BD^{-1}C & B \\ C - D^{-1}DC & D \end{bmatrix}$$

$$= \det\begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D)$$

Gauss Elimination:

- row operations: · swap two rows, put (-1) in front
  - multiple one row, add the result to another row

$$\lambda \cdot r_i + r_j \overset{\text{put in}}{\rightsquigarrow} r_j$$

- similarly for column operation

Goal: make the matrix upper triangular

Recall: ① "Algebra-like object" Field K

② "Modulo-like obj" Vector space

• What is a field?

a field is a set K with $+, \cdot, 0, 1, i$

such that:

• $(K, +, 0)$ every element in K has an inverse

$\forall x, \exists x' \in K$, s.t. $x' + x = 0$

• $(K^* = K \setminus \{0\}, \cdot, 1)$. $\forall y \in K^*, \exists y' \in K^*$ $y \cdot y' = 1$

Ex: • $\mathbb{R}$ $\quad \mathbb{R}^* = \mathbb{R}_{>0} \cup \mathbb{R}_{<0}$

• $\mathbb{C}$

• $\mathbb{Q}$

• p prime number

$\mathbb{F}_p = \mathbb{Z}/p\cdot\mathbb{Z} = \{0, 1, \dots, p-1\}$

Ex: $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$

Q: Does every element in $\mathbb{F}_5$ have an "+" inverse

$2+3=0 \quad 1+4=0$

Q: $1 \cdot \underline{1} = 1 \quad 2 \cdot \underline{3} = 1 \quad 3 \cdot \underline{2} = 1 \quad 4 \cdot \underline{4} = 1$

①: Does $\mathbb{Z}/6\cdot\mathbb{Z}$ form a field under the natural $(+, \cdot)$

No. $\quad 2 \cdot ? = 1$

$n \geq 1$ integer $\underset{\text{exists unique}}{\exists !}$ $\mathbb{F}_{p^n}$ finite field with p^n many elements

Let $p(x) \in \mathbb{F}_p[x]$

with variable x with coefficient in $\mathbb{F}_p$

Ex: $p=5$ $\quad 3x^2+4x-2 \in \mathbb{F}_5[x]$

Assume $f(x)$ is irreducible in $\mathbb{F}_p[x]$ i.e. $f(x) \neq g(x)h(x)$ with h & g of smaller degree

Ex: $p=5$ $\quad f(x) = x^2+1 = (x-2)(x-3)$ in $\mathbb{F}_5$

$= x^2 - 5x + 6$

$= x^2 + 1 - 5(x-1)$

$= x^2 + 1$

$f(x) = x^2 + 2$

then $\mathbb{F}_p[x] / (f(x) \cdot \mathbb{F}_p[x])$ is a field of size $p^{n(\text{degree of } f)}$

$\mathbb{F}_{5^2} = \mathbb{F}_5[x] / (x^2+2)$ typical element: $x, 3, x^3+1$

$x^2+2=0$

$5=0$

$x^2+3 = (x^2+2)+1 = 1 \qquad x^2 = -2, \; x^3 = -2x$

$\mathbb{F}_5[x]/(x^2+2) = \{ax+b \mid a, b \in \mathbb{F}_5\} \simeq (\mathbb{F}_5)^2$ as vector space

Does every non-zero element in $\mathbb{K}$ has inverse

$\mathbb{C} \simeq \mathbb{R}^2$

$\mathbb{Q}[\sqrt{-1}] := \mathbb{Q}[x]/(x^2+1) \simeq \mathbb{Q}^2$

$= \{ax+b \mid a, b \in \mathbb{Q}\}$

$(ax+b)(cx+d) = (-ac+bd)+(bc+ad)x \quad (acx^2 = ac(x^2+1-1))$

Let K be a field

A vector space V over K is a set such that:

① $(V, +, 0)$ is an abelian group (same as ...)

② $K \curvearrowright V$ (scalar product)

• a function (map) on a set S, valued in K is an assignment to each element in S, some element in K

$\mathrm{Map}(S, K) = K^S$

• The set of all functions from S to K is

Ex: if $S = \{*\}$ (1 point) $\quad$ if $S = \{a, b\}$

$K^S = K \quad\quad K^S = K^2$

$S = \emptyset \quad K^{\emptyset} = K^0 = \{0\}$

$f, g \in \mathrm{Map}(\{1, 2\}, K)$ (vector space)

$f, g : \{1, 2\} \longrightarrow \mathbb{R}$

• $(f+g)(x) = f(x) + g(x)$

• $\forall c \in K, \ (c \cdot f)(x) = c f(x)$

· $S = \{1, 2, \dots, n\}$ $K^S = K^n$ | $K = R, C, Q$ field

$S = [n] \times [m]$

$= \{(i,j) \mid \begin{smallmatrix} 1 \le j \le n \\ 1 \le j \le m \end{smallmatrix}, \text{integer}\}$

$K^S = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \middle| a_{ij} \in K \right\}$ vector space

· Let V be a vector space over K

A subspace $W \subset V$ is a subset that is closed under + and K

Ex: $K \in R$

$V = R^R = \text{Map}(R, R)$ all functions on R

$W = R[x] = \{a_n x^n + \dots + a_0 \mid a_i \in R, n \in Z_{\ge 0}\}$

Category Theory

· a set of objects

· morphisms between objects

· composition of morphisms

Ex 1: set: the category of sets

· objects: any set

· morphisms: Given S_1, S_2 two sets, $\text{mor}(S_1, S_2) = \text{Map}(S_1, S_2)$

· compositions: $\underset{f_{12}}{\text{mor}(S_1, S_2)} \times \underset{f_{23}}{\text{mor}(S_2, S_3)} \to \text{Mor}(S_1, S_3)$

$f_{23} \cdot f_{12} = f_{23}(f_{12}(\ ))$

Vect_K: the category of vector spaces / K

- an object is a vector space over K
- a morphism between V_1, V_2 is a linear map

$\text{Mor}(V_1, V_2) = \text{LinMap}(V_1, V_2)$?

Note: the set $\text{LinMap}(V_1, V_2)$ also forms a vector space

- Let $f: V \to W$ be a linear map

? - $\ker(f) = \{ v \in V \mid f(v) = 0 \} \subset V$

- $\text{im}(f) = \{ w \in W \mid \exists v \in V, w = f(v) \} \subset W$
- $\text{coker}(f) = \frac{W}{\text{im}(f)}$

Recall: A vector space V over a field K is a set V such that

- $+ : V \times V \to V$
- $\cdot : K \times V \to V$

are well-defined and compatible

Given 2 vector spaces, V, W

LinMap (V, W) is the set of linear maps

We also write it as $\mathrm{Hom}(V, W)$

homomorphism

- $\mathrm{Hom}(V, W)$ also forms a vector space over K
- Direct Sum:

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\} = V \times W$$

$(v_1, w_1), (v_2, w_2) \in V \oplus W$ · $c \in K$

$c(v, w) = (cv, cw)$

$(v_1, w_1) + (v_2, w_2)$

$= (v_1 + v_2, w_1 + w_2)$

Ex: $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$

- Quotient space:

Let V be a vector space and $W \subset V$ a subspace then V/W is the vector space consisting of equivalence class in V, where we say $v_1, v_2 \in V$ are equivalent if $v_1 - v_2 \in W$

$$V/W = v + W, \quad v \in V$$

Ex: $V = \mathbb{R}^2$

$W = \{(0,y) \mid y \in \mathbb{R}\}$

$v_1 = (-1,1)$ $v_2 = (-1,-1)$ $v_1 - v_2 = (0,2) \in W$

we say $v_1 \sim v_2$

the line $v_1 + W = \{v_1 + w \mid w \in W\}$ is the equivalence class that v_1 belongs to

Let $v_1 + W, v_2 + W \in V/W$

then $(v_1 + W) + (v_2 + W)$

$= (v_1 + v_2) + W$

This is well defined cuz

if $\tilde{v}_1 + W = v_1 + W$ i.e. $\tilde{v}_1 \in v_1 + W$

? $\tilde{v}_2 + W = v_2 + W$ $\tilde{v}_2 \in v_2 + W$

addition of two elements from equivalence class

then $\tilde{v}_1 + \tilde{v}_2 \in v_1 + v_2 + W$

if $v_1 - \tilde{v}_1 \in W$, $v_2 - \tilde{v}_2 \in W$ then

$(v_1 + v_2) - (\tilde{v}_1 + \tilde{v}_2) \in W$

π: $V \to V/W$

$v \mapsto v + W$?

If V is a $\mathbb{R}$ vector space and has inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$

then given $W \subset V$, we can form $W^\perp \subset V$

recall that, given $\langle -, - \rangle$ $v_1 \perp v_2 \iff \langle v_1, v_2 \rangle = 0$

- $v_1 \perp W \iff \forall w \in W, \langle v_1, w \rangle = 0$

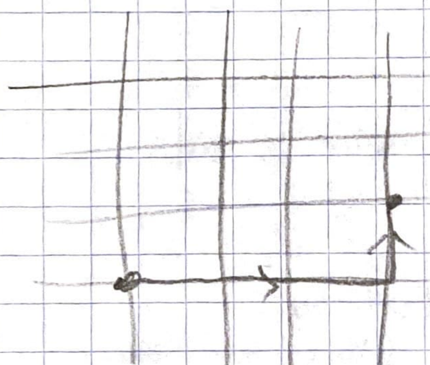
$W^\perp = \{v \in V \mid v \perp W\} \subset V$ *definitions

$\pi : V \to V/W$ quotient map

π restricted to $W^\perp$ gives a bijection

$\pi|_{W^\perp} : W^\perp \to V/W$

Aside on distance:



l_1: $d_1((x_1, x_2, \dots, x_n), (y_1, \dots, y_n))$

l_2: $\left[\sum_{i=1}^{n} |x_i - y_i|^2\right]^{\frac{1}{2}}$

$\dim V/W = \dim V - \dim W$

Dual space:

V vector space ~~over~~ $/\mathbb{K}$

$V^* = \mathrm{Hom}(V, \mathbb{K})$

= the set of linear functions on V

Ex: $V = \mathbb{R}^2$ x_1, x_2

$V^* = \{ax_1 + bx_2 \mid a, b \in \mathbb{R}\}$

$x_1 \in V^* \ni x_2$

Ch3. 1. Basis.

span linear independence

Def: a collection (possibly infinite) Let V be a v.s. over K
of vectors $v_1, v_2, \dots$ is called a basis
if any $v \in V$ can be written uniquely as
finite linear combination of $\{v_i\}$

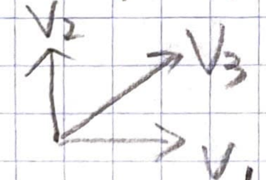
Ex: $K[x]$ vector space of 1-variable polynomials
$\{1, x, x^2, \dots\}$ forms a basis.

· Span: Let $v_1, v_2, \dots, v_n$ be vectors in V,
then we have a map

$$f: K^n \to V$$
$$(a_1, \dots, a_n) \mapsto a_1v_1 + a_2v_2 + \dots + a_nv_n$$

$\mathrm{Span}\{v_1, \dots, v_n\} = \text{image of } f = \{a_1v_1 + \dots + a_nv_n \mid a_i \in K\}$

We say $\{v_1, \dots, v_n\}$ span V
if $\mathrm{span}\{v_1, \dots, v_n\} = V$

Ex: 

$V = \mathbb{R}^2$
$\mathrm{span}(v_1, v_2, v_3) = V$? (v_1, v_2)

We say $\{v_1, \dots, v_n\}$ is linearly independent, if the
map $f: K^n \to V$ is injective ?
if $(a_1, a_2, \dots, a_n) \neq (0, 0, \dots, 0)$, then $a_1v_1 + \dots + a_nv_n \neq 0$

$$V = \mathbb{R}^3$$

$\mathrm{span}(v_1, v_2, v_3) = XZ$ plane

$= \{(a, 0, b) \mid a, b \in \mathbb{R}\}$

$v_2 = (2,0,0)$

$v_1 = (1,0,0)$

$v_3 = (0,0,1)$

$\{v_1, v_2\}$ is not linearly independent

because $2 \cdot v_1 + (-1) \cdot v_2 = 0$

?

$\cdot \{v_1, v_3\}$ is lin indep

- $\dim_{\mathbb{K}} V$ = # of elements in a basis
- any collection of linearly indep vectors in V

  $\{v_1, \dots, v_m\}$

  can be completed to a basis

Ex: Let $v_1 = (1, 1, 0)$ $v_3 = (0, 1, 1)$

$v_2 = (1, 0, 1)$ in $\mathbb{R}^3$

Q_1: are they linearly independent

$$\underbrace{\begin{array}{ccc} v_1 & v_2 & v_3 \\ \left(\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}\right. & & \left.\right) \end{array}}_{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det A = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

is A invertible $\Leftrightarrow \det A = 0$

$$A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$A^{-1} A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A^{-1}(0)$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

basis lin indep V

span V b

Let $V_4 = (1,1,1)$

Q: are $\{v_1, v_2, v_3, v_4\}$ linearly indep

A: No Cuz $\{v_1 v_2, v_3\}$ forms a basis

$\therefore V_4 = a_1 v_1 + a_2 v_2 + a_3 v_3$

$T: V \to W$ $\quad$ V, W fin. dim.

$\operatorname{rank}(T) := \dim(\operatorname{im} T)$ $\quad$ $\operatorname{im} T = T(V)$

$\Rightarrow \operatorname{rank}(T) \leq \dim W$ $\quad$ $\operatorname{rank}(T) \leq \dim V$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, two heads, "\pi", "\text{surjection}"'] & W \\ V/\ker(T) \arrow[r, "\simeq"] & \operatorname{im}(T) \arrow[u] \end{tikzcd}$$

$w \in \operatorname{im}(T)$

$T^{-1}(w)$ is invariant under translation by $\ker(T)$

$\Leftrightarrow \forall v \in T^{-1}(w), \tilde{v} \in \ker(T)$

$v + \tilde{v} \in T^{-1}(w) \Leftrightarrow T(v+\tilde{v}) = w$

$\hookrightarrow$ injection

$\rightarrowtail$

$\forall v_1, v_2 \in T^{-1}(w)$?

$v_1 - v_2 \in \ker(T)$

? $\operatorname{rank}(T) = \dim(\operatorname{im} T) = \dim(V/\ker T)$

Let T be a rank r map
from V to W, then there exists basis
$v_1, \cdots, v_n$ of V $w_1, \cdots, w_m$ of W
s.t. w.r.t. basis T is $\left[\begin{array}{c|c} \ddots & 0 \\ \hline 0 & 0 \end{array}\right]$

$T(v_1) = w_1$
$\vdots$
$T(v_r) = w_r$
$T(v_{r+1}) = 0$
$\vdots$
$T(v_n) = 0$