① LPU decomposition

Let M be a $n\times n$ invertible matrix, then M can be written as a product of the following form:

$$M = L \cdot P \cdot U$$

L = lower triangular matrix $\begin{pmatrix} * & & 0 \\ * & * & \\ * & & \ddots \\ & & & * \end{pmatrix}$

U = upper triangular matrix $\begin{pmatrix} & \equiv \\ 0 & \end{pmatrix}$

P = permutation matrix

Given a bijection $\sigma = \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$

i.e. $\sigma \in S_n$ $(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{else} \end{cases}$

Ex: $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

- left multiplication by P_σ permute the rows
- right multiplication by P_σ permute the cols

- the set of $n\times n$ invertible matrices, denoted as $G = GL(n)$ forms a group

i.e. ① $\forall g_1, g_2 \in G$. $g_1 \cdot g_2 \in G$ $(g_1 \circ g_2)\circ g_3 = g_1 \circ (g_2 \circ g_3)$

② $\exists\, e \in G$, identity element
$e \cdot g = g \cdot e = g$

③ $\forall g \in G$, $\exists\, g^{-1} \in G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$

- Subgroup: Let G be a group, a subset $H \subset G$ is a subgroup if H satisfies ①, ②, ③

Ex: Let B_- denote the set of lower triangular matrices

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 = r_2 - 2r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 = r_3 - 3r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{array}\right) \xrightarrow{r_3 = r_3 - 5r_2}$$ the inverse is on the right

if we're given a lower triangular matrix, then if we modify $r_i = r_i + c \cdot r_j$ for some $j<i$ then the new matrix is still lower triangular

$$\begin{array}{c c} & j \\ i \to & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & c & & 1 \\ & & & & 1 \end{pmatrix} \cdot L \end{array}$$

- Similarly, the set of all invertible upper triangular matrices, called B^+, forms a subgroup of $GL(n)$
- In general, given $H_1, H_2 \subset G$ subgroups, we can consider the subset

$$H_1 \cdot H_2 = \left\{ h_1 \cdot h_2 \,\middle|\, \begin{matrix} h_1 \in H_1 \\ h_2 \in H_2 \end{matrix} \right\} \subset G$$

- $G = GL(n)$

$B_- \cdot B_+ \subset G$ $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \cdot \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is it everything ✗ (does not stand)

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ ✗ $\begin{pmatrix} - & 0 \\ - & - \end{pmatrix} \begin{pmatrix} - & - \\ 0 & - \end{pmatrix}$

Given an arbitrary 2x2 invertible matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

We want to use B_- multiplication on the left to "simplify" it

i.e. we can do row rescaling

$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot M$ changes the first row by a factor of a.

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ cr_1 + r_2 \end{pmatrix}$$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

① if $a \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{r_1 = \frac{1}{a} r_1} \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} \xrightarrow{r_2 = r_2 - c r_1} \begin{pmatrix} 1 & b/a \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

$\rightsquigarrow \begin{pmatrix} 1 & b/a \\ 0 & \end{pmatrix}$

if $a=0$

$$M = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & d/c \end{pmatrix}$$

with left multiplication by $\overset{B_-}{\begin{pmatrix} \varepsilon & 0 \\ \varepsilon & \end{pmatrix}}$, we have

$$M = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & d/c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$$

· $G = B_- \cdot P \cdot B_+$ $\quad G \supset B_- B_+$

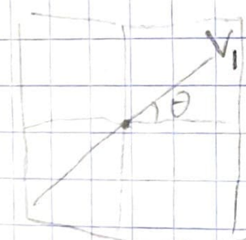
$L^{-1} L \cdot M = L^{-1} P U \quad M = L^{-1} P \cdot U \quad B_- = L$, $B_+ = U$

· Flag variety (complete)

· A flag in a n-dim vector space V is an increasing sequence of subspaces

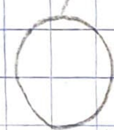
$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \qquad \dim V_k = k$$

Ex: $V = \mathbb{R}^2$



$\theta \in [0,\pi]/(0\sim\pi) \overset{\text{a set}}{\simeq} [0,\pi) = \mathbb{R}/\mathbb{Z}\pi$

$V_0 \subset V_1 \subset V_2$, $V_0 = 0$, $V_2 = V$

$\simeq$  $= \mathbb{RP}^1$

$Fl_2(\mathbb{R}) = \mathbb{RP}^1$

· Let Fl_n be the set of flags in

· $V = \mathbb{R}^3$ $\quad Fl_3(\mathbb{R}^3)$

to specify a line in $\mathbb{R}^3$

S^2 (2-dim sphere) $/\{\pm\}$

$$S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$$

Ex:

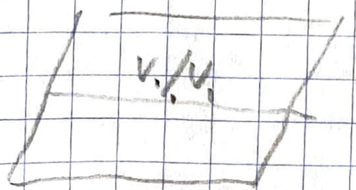
to parameterize $V_1 \subset \mathbb{R}^3$, we need a point $S^2/\{\pm\}$

data of $\mathbb{RP}^2$

to parameterize V_2, s.t. $V_1 \subset V_2 \subset \mathbb{R}^3$

it is equivalent to specify $V_1/V_1 \subset V_2/V_1 \subset \mathbb{R}^3/V_1$

data of a $\mathbb{RP}^1$



$$Fl_2 \hookrightarrow Fl_3 \qquad V_0 \subset V_1 \subset V_2 \subset V_3$$

$$\begin{array}{c} Fl_3 \\ \downarrow \\ \mathbb{P}^2 \end{array} \qquad V_0 \subset V_1 \subset V_3$$

$$Fl_n \longrightarrow \mathbb{P}^{n-1}$$

$$Fl_n = \{[\overset{0}{\overset{\shortparallel}{V_0}} \subset V_1 \subset \cdots \subset \overset{V}{\overset{\shortparallel}{V_n}}] \mid \dim V_i = i\}$$

$$\mathbb{P}^{n-1} = \{[\overset{0}{\overset{\shortparallel}{V_0}} \subset V_1 \subset \overset{V}{\overset{\shortparallel}{V_n}}] \mid \dim V_i = i\}$$

- If we're given a basis of V $e_1, e_2, \cdots, e_n$ then we can produce a flag

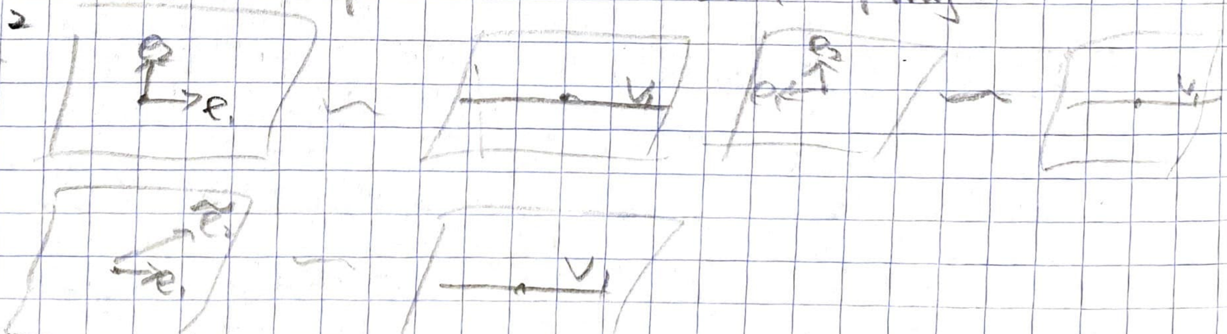
$V_1 = \text{span}\langle e_1 \rangle$

$V_2 = \text{span}\langle e_1, e_2 \rangle$

$\vdots$

But different basis can produce the same flag

Ex. $V = \mathbb{R}^2$



(e_1, e_2) and $(\tilde{e}_1, \tilde{e}_2)$ generate the same flag. if $\tilde{e}_1 = a \cdot e_1$

$\tilde{e}_2 = b \cdot e_1 + c \cdot e_2$

$$(\tilde{e}_1, \tilde{e}_2) = (e_1, e_2)\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$Fl_n = GL(n) / B_+(n)$

$\cdot\ g \in GL_n \rightsquigarrow e_1, \dots, e_n$ as column vectors of g

we say $g \sim \tilde{g}$ if $g = \tilde{g} \cdot b \leftarrow$ upper triangular

$Fl_n = \{$ set of basis $e_1, \dots, e_n / (\ )\}$

$= GL(n) / B_+$

· inner product on a 2-dim$_{\mathbb{R}}$ space

$V = \mathbb{R}^2$

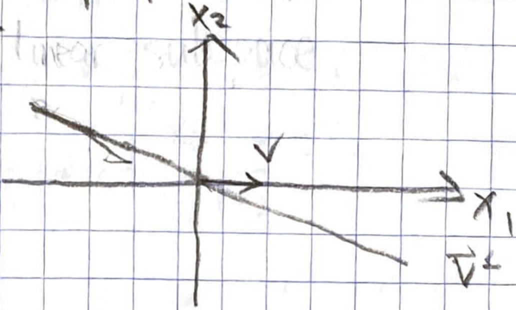
· a funny inner product $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := (x_1, x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

(quadratic form, a.k.a. symmetric bilinear form)

We can ask: for a given vector $\vec{v} \in V$ what is the set of all vectors perpendicular to $\vec{v}$

$\{ w \in V \mid \langle w, v \rangle = 0 \} =: \vec{v}^{\perp}$

↑ a linear subspace



$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}^{\perp} =$

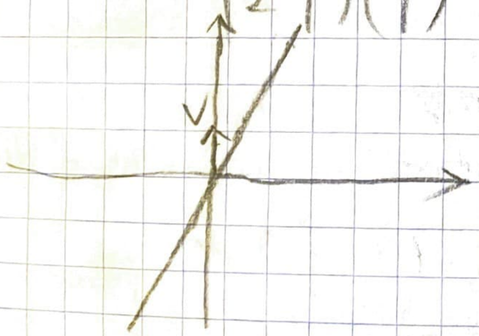
$w = (x_1, x_2)$

$(x_1, x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \iff (x_1, x_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \quad x_1 + 2x_2 = 0$

$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\vec{v}^{\perp} =$

$(x_1, x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \iff (x_1, x_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$

$\iff 2x_1 + x_2 = 0$



$B: \vec{V} \times \vec{V} \to \mathbb{K}$ is bilinear

$B(a\vec{v} + b\vec{w}, \vec{u})$
$= B(a\vec{v}, \vec{u}) + B(b\vec{w}, \vec{u})$
$= a \cdot B(\vec{v}, \vec{u}) + b \cdot B(\vec{w}, \vec{u})$

· Let $\mathbb{K}$ be a field, V be a fin. dim. vector space $/\mathbb{K}$

Let $B: V \times V \to \mathbb{K}$

be a symmetric bilinear form $B(v,w) = B(w,v)$

This data is equivalent to a quadratic form

$Q: V \to \mathbb{K}$

by setting $Q(v) = B(v,v)$

Conversely $B(v,w) = \frac{1}{4}(B(v+w, v+w) - B(v-w, v-w))$

$= \frac{1}{2}(B(v+w, v+w) - B(v,v) - B(w,w)) = \frac{1}{4}(Q(v+w) \ldots$

Given a basis $e_1, \dots, e_n$ of V and a bilinear form B we can produce a matrix

$B_{ij} = B(e_i, e_j)$

Since B is symmetric, $B(e_i, e_j) = B(e_j, e_i)$

$\therefore B_{ij} = B_{ji}$

If $v = v_1 e_1 + \dots + v_n e_n$, $v_i \in K$

$w = w_1 e_1 + \dots + w_n e_n$, $w_i \in K$

then $B(v,w) = B\left(\sum_{i=1}^{n} v_i e_i, \sum_{j=1}^{n} w_j e_j\right)$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} B(v_i e_i, w_j e_j) = \sum_{i,j} v_i w_j B_{ij}$

$= (v_1 \dots v_n) \begin{pmatrix} B_{11} & B_{12} & \dots \\ \vdots & & \\ \vdots & \dots & \dots \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

- A basis $\{e_i\}_{i=1}^{n}$ of V is orthogonal w.r.t. bilinear form B if for $i \neq j$, we have $B(e_i, e_j) = 0$, i.e., $[B_{ij}]$ matrix is diagonal

Lemma: every bilinear form B on a finite dim V admits an orthogonal basis

Pf: if $B = 0$, then any basis works

- we induct on $\dim_K V$ suppose the statement is true for $\dim_K V \leq n-1$ any v.s. V, B

consider any v.s. V and a bilinear form B on V s.t. $\dim V = n$

- if $B = 0$, nothing needs to be proven
- if $B \neq 0$, then $\exists$ a vector $v \in V$ s.t. $B(v,v) \neq 0$ $Q(v)$

Let $W = v^{\perp B} = \{w \in V \mid B(w, v) = 0\}$

Then $v \notin W$

claim: $\dim W = n-1$

because consider

$T: V \to K$ $w \mapsto B(w,v)$ and $\operatorname{im}(T) = K$ ($\because T(v) \neq 0$)

$W = \ker(T)$

we know

$\dim V = \dim(\operatorname{im}(T)) + \dim \ker(T)$

$\dim(\operatorname{im} T) = 1$

$\therefore \dim(\ker(T)) = n-1$

$\dim W = n-1$

By induction hypothesis, for (W, B),
the statement holds. Thus, there exists a basis of W,
$\{e_1, \dots, e_{n-1}\}$, that is orthogonal w.r.t. B.

Let $e_n = v$, then $\{e_1, \dots, e_{n-1}, e_n\}$ forms a basis of V

Let $\{e_1, \dots, e_n\}$ be an orthogonal basis of B

$$[B_{ij}] = \begin{bmatrix} B_{11} & & 0 \\ & \ddots & \\ 0 & & B_{nn} \end{bmatrix}$$

If $\mathbb{K} = \mathbb{R}$, then we can reorder the basis, s.t.

$\underbrace{B_{11}, \dots, B_{pp} > 0}_{p \text{ many}}$, $\underbrace{B_{p+1,p+1}, \dots, B_{q+p,q+p} < 0}_{q \text{ many} < 0}$

$$[B_{ij}] = \begin{bmatrix} + & & & & & \\ & + & & & & \\ & & - & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$ we call such B of signature (p, q)

By rescaling $\{e_i\}$, we can make $(B_{ij}) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$

$\mathbb{K} = \mathbb{C}$ Suppose $B(e_i, e_i) = \lambda \in \mathbb{C}$, $B(e_i, e_j) = 0$ for $i \neq j$

$$[B_{ij}] = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & 0 \\ & & & & 0 \end{pmatrix}$$

let $\tilde{e}_i = \frac{1}{\sqrt{\lambda_i}} e_i$, for $1 \le i \le r$

$\tilde{e}_i = e_i$ for $i > r$

r is also the rank

$B(\tilde{e}_i, \tilde{e}_i) = \left(\frac{1}{\sqrt{\lambda_i}}\right)^2 \cdot B(e_i, e_i) = \frac{1}{\lambda_i} \lambda_i = 1$

- Let B be a bilinear form on V
- Let $\{e_1, \dots, e_n\}$ be a basis of V
- Let $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ be another basis of V
- we have $(e_1, \dots, e_n) = (\tilde{e}_1, \dots, \tilde{e}_n) \cdot A$

$(e_1, \dots, e_n) \cdot A^{-1} = (\tilde{e}_1, \dots, \tilde{e}_n)$ for some invertible matrix

$$e_i = (\tilde{e}_1, \dots, \tilde{e}_n) \begin{pmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{pmatrix} = \sum_{j=1}^{n} \tilde{e}_j A_{ji}$$

$$\tilde{e}_i = (e_1, \dots, e_n) A^{-1}_{i} = \sum_{j=1}^{n} e_j (A^{-1})_{ji}$$

$B_{ij} = B(e_i, e_j) \quad \tilde{B}_{ij} = B(\tilde{e}_i, \tilde{e}_j)$

Express $\{e_i\}$ using $\{\tilde{e}_i\}$ we have

$$B_{ij} = B(e_i, e_j) = B\left(\sum_{a=1}^{n} \tilde{e}_a A_{ai}, \sum_{b=1}^{n} \tilde{e}_b A_{bj}\right)$$

$$= \sum_{a,b} A_{ai} A_{bj} B(\tilde{e}_a, \tilde{e}_b)$$

$$= \sum_{a,b} A_{ai} A_{bj} \tilde{B}_{ab} = \sum_{a,b} (A^t)_{ia} \tilde{B}_{ab} A_{bj}$$

$$= [A^t \tilde{B} A]_{ij}$$

$$B = A^t \tilde{B} \cdot A$$

basic moves:

- swap i & j rows & i&j columns

$$B \rightsquigarrow S_{ij}^t B S_{ij} \qquad S_{ij} = \begin{array}{c} \quad i \quad j \\ \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \end{array}$$

- change i-th row by adding a multiple of j-th row

change i-th column

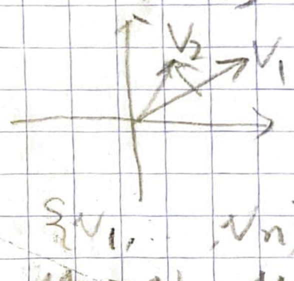
$r_i = r_i + \lambda \cdot r_j \qquad r_i$ i-th row

$c_i = c_i + \lambda c_j \qquad c_i$: i-th column

$r_i \to \lambda r_i \quad c_i \to \lambda c_i \quad \lambda \in \mathbb{K}^*$

Gram-Schmidt

$\{v_1, \dots, v_n\} \to$ orthogonal basis $\{u_1, \dots, u_n\}$



$\vec{v}_2 = a\vec{v}_1 + \beta \vec{v}_1^{\perp}$

$$P_v(u) = \frac{\langle v, u\rangle}{\langle u, u\rangle}$$

$\{v_1, \dots, v_n\} \ \{u_1, \dots, u_n\}$

$u_1 = v_1 \quad u_2 = v_2 - P_{v_2}(u_1) \cdot u_1$

$u_3 = v_3 - P_{v_3}(u_2) u_2 - P_{v_3}(u_1) u_1$

$$u_n = v_n - \sum_{i=1}^{n-1} P_{v_n}(u_i) \cdot u_i$$

Quadratic Form & Symm Bilinear Form

- V: vector space over $\mathbb{K}$
- $B: V \times V \to \mathbb{K}$ $\quad B(x,y) \in \mathbb{K} \quad x,y \in V$
  $\qquad\qquad\qquad\qquad B(x,y) = B(y,x)$ symmetric
  $\qquad\qquad\qquad\qquad B(x_1+x_2,y) = B(x_1,y)+B(x_2,y)$

$Q: V \to \mathbb{K}$

- $Q(x) = B(x,x)$ $\quad$ from $B = Q$
- $B(x,y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$ $\quad$ from Q to B

Lemma: for any V, and B symmetric bilinear form
$\exists$ a basis $e_1, \cdots, e_n$ of V, s.t. $e_i \perp_B e_j$ w.r.t. to B
$B(e_i, e_j) = 0$ if $i \neq j$

- Given a basis $e_1, \dots, e_n$ of V
  we define matrix $[B]$ with entries $B_{ij} = B(e_i, e_j)$
  Then $B_{ij} = B_{ji}$ $\quad [B] = [B]^t$

  if $v = c_1 e_1 + \cdots + c_n e_n$
  $\quad w = d_1 e_1 + \cdots + d_n e_n$
  then $B(v,w) = (c_1, \cdots, c_n)[B]\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$

If we change basis to $\tilde{e}_1, \dots, \tilde{e}_n$
$(e_1, \cdots, e_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)A$
and define $\tilde{B}_{ij} = B(\tilde{e}_i, \tilde{e}_j)$
$[B] = A^t[\tilde{B}]A$

- A symmetric bilinear form B on V is an inner product (non-degenerate) if for any vector $0 \neq v \in V$, $\exists$ some $w \in V$, s.t. $B(v,w) \neq 0$
  $\Leftrightarrow B(v, -): V \to \mathbb{K}$ is non-zero
- When we diagonalize a non-degen bilinear form
  $[B] \rightsquigarrow [\tilde{B}] = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $\quad \lambda_i \neq 0$

If $\mathbb{K}=\mathbb{R}$, an inner product on V is a bilinear form B s.t. $B(v,v)>0$ for all $v\in V$ symmetric
$\neq 0$

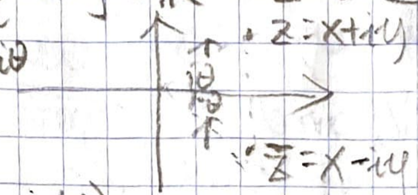
i.e. in diagonal form, one can find $[\tilde{B}]=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$

$B: V\times W\to\mathbb{K} \Leftrightarrow \Phi: V\to W^* \Leftrightarrow \Psi: W\to V^*$

- Recall:
  - a complex conjugation on $\mathbb{C}$

$z = x+iy = re^{i\theta}$ $x,y\in\mathbb{R}$ $z\in\mathbb{C}$

$\bar{z} = x - i\cdot y = re^{-i\theta}$



$z\cdot\bar{z} = (x+yi)(x-iy)$ $\quad \bar{z}z = r e^{i\theta}\cdot r e^{-i\theta}$
$= x^2-(yi)^2$ $\quad = r^2$
$= x^2+y^2$

- Let $S: V\times V\to\mathbb{C}$

such that "S is $\mathbb{C}$-linear in the 2nd slot, and anti $\mathbb{C}$-linear in the first slot"

- $S(v, aw_1+bw_2) = aS(v,w_1)+bS(v,w_2)$ $\forall a,b\in\mathbb{C}$

- $S(av_1+bv_2, w) = \bar{a}S(v_1,w)+\bar{b}S(v_2,w)$

Ex: $\mathbb{C}^n$ $v=(z_1,\dots,z_n)\in\mathbb{C}^n$ $w=(u_1,\dots,u_n)\in\mathbb{C}^n$ $z_i\in\mathbb{C}$

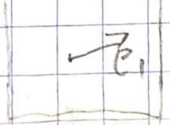
we can define the standard ses-linear form
$S(v,w)=\bar{z}_1\cdot u_1+\bar{z}_2\cdot u_2+\dots+\bar{z}_n\cdot u_n$

- ~~Her~~ $S(w,v)=\overline{S(v,w)}$

- A sesquilinear form S is hermitian if $S(v,w)=\overline{S(w,v)}$

If V is a vector space over $\mathbb{C}$, and $v\in V$, then it doesn't make sense to say $\bar{v}$

Eg. Let V be a 1-$\dim_{\mathbb{C}}$ v.s. over $\mathbb{C}$



then $v=c_1 e_1$, $c_1\in\mathbb{C}$
suppose we define $\bar{v}=\bar{c}_1 e_1$, then this definition

depends on the choice of e_1

say $\tilde{e}_1 = e^{i\theta} \cdot e_1$, ~~$v =$~~ $\tilde{c}_1 \cdot \tilde{e}_1$

if we try to define $\bar{v} = \overline{\tilde{c}_1} \cdot \tilde{e}_1$, then we get a contradiction

If S is a hermitian sesquilinear form
$S: V \times V \to \mathbb{C}$
then knowing its value on $S(v,v)$ is enough to recover $S(v,w)$

Let $H: V \times V \to \mathbb{C}$ be a hermitian form
$v, w \in V$, we say $v \perp w$ w.r.t. H if $H(v,w) = 0$
$v \perp w \Leftrightarrow w \perp v$

Lemma: Given any hermitian form H, there exists an orthogonal basis $e_1, \dots, e_n$ of V, i.e. $e_i \perp e_j \ \forall i \neq j$

proof / algorithm:
. If $\dim V = 0$, then done
. If $H \equiv 0$, then pick any basis of V
. Otherwise, pick a $v \in V$, s.t. $H(v) \neq 0$
Let $V' = v^\perp = \{w \in V \mid w \perp v\}$
repeat the process replacing V by V'.

Ex: $V = \mathbb{C}$
$S: V \times V \to \mathbb{C}$
$(z, w) \mapsto e^{i\theta} \cdot \bar{z} \cdot w$
non-hermitian sesquilinear form
$\mathbb{C}^n$
$S(\vec{z}, \vec{w})$
$\sum_{i=1}^{n} c_i \cdot \bar{z}_i \cdot w_i$
$c_i \in \mathbb{C}$
$S(\lambda \vec{z}, \vec{w}) = \bar{\lambda} S(\vec{z}, \vec{w})$
$S(\vec{z}, \lambda \vec{w}) = \lambda S(\vec{z}, \vec{w})$
if some $c_i \notin \mathbb{R}$, then
$S(\vec{z}, \vec{w}) \neq \overline{S(\vec{w}, \vec{z})}$

Let H be a Hermitian form on V
$e_1, \dots, e_n$ be basis of V
then we can define $H_{ij} := H(e_i, e_j) \in \mathbb{C}$
$[H_{ij}]$ $n \times n$ matrix
and $H_{ij} = \overline{H_{ji}}$
any complex matrix A
we can define $A^\dagger := \overline{A^t}$
if $A = A^\dagger$, then A is a hermitian matrix
If $[H_{ij}]$ is diagonal $\Rightarrow H_{ii} \in \mathbb{R}$

Cor: Given H: hermitian form on V
$\exists$ basis $\{e_1, \dots, e_n\}$ of V.
s.t. $H(e_i, e_j) = \begin{cases} 0 & i \neq j \\ +1, -1, \text{or } 0 & i = j \end{cases}$

· Let $V = \mathbb{C}^2$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$[H] = \begin{pmatrix} 2 & i+1 \\ 1-i & 3 \end{pmatrix}$$

Q: find a new basis $(\tilde{e}_1, \tilde{e}_2)$ s.t.
$[\tilde{H}_{ij}]$ is diagonal

$$\begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$$

do column operation $C_2 \leftarrow C_2 - C_1 \cdot \frac{1+i}{2}$

do row operation $r_2 \leftarrow r_2 - r_1 \overline{\left(\frac{1+i}{2}\right)}$

$\Big\downarrow$ col. op.

$$\begin{pmatrix} 2 & 0 \\ 1-i & 2 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Recall:
· complete flag in a v.s. V
Def a flag in V is "increasing" sequence of vector spaces
$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \quad \dim V_i = i$

Given a basis $\{e_1, \ldots, e_n\}$ of V
→ obtain the standard flag associated with the basis
$V_1 = \mathrm{span}\{e_1\}$ $V_2 = \mathrm{span}\{e_1, e_2\}$ ... $V_k = \mathrm{span}\{e_1, \ldots, e_k\}$

Given a flag $V_\cdot$ in V and a basis $\{e_i\}$, we say the basis is adapted to the flag
if $V_k = \mathrm{span}\{e_1, \ldots, e_k\}$

Lemma:
Given a flag $V_\cdot$, there exists a basis $\{e_i\}$ adapted to $V_\cdot$

Pf: · pick any $e_1 \in V_1$, s.t. e_1 is a basis of V_1
· pick any vector $e_2 \in V_2$, $e_2 \notin V_1$
then $V_2 = \mathrm{span}\langle e_1, e_2 \rangle$
· pick $e_k \in V_k \setminus V_{k-1}$

Suppose we have certain notion of orthogonality:
Ex 1: a given symmetric bilinear form $B: V \times V \to K$,
we say $v \perp w$ if $B(v, w) = 0$
Ex 2: · Given a Hermitian form
$H: V \times V \to \mathbb{C}$ $\quad v \perp w$, $H(v, w) = 0$

Then given a flag $V_\cdot$, we can construct orthogonal basis adapted to $V_\cdot$. Assume the Hermitian form is positive definite

Def: Let V be a v.s. $/\mathbb{C}$
an H is positive definite
if $\forall\, 0 \neq v \in V$, $H(v, v) >$

Let $\tilde{e} \in V_2 \setminus V_1$
suppose $H(\tilde{e}_2, e_1) \neq 0$
then we may define $e_2 = \tilde{e}_2 - \lambda e_1$
such that $H(e_2, e_1) = 0$
$H(\tilde{e} - \lambda e_1, e_1) = 0$
$\Leftrightarrow H(\tilde{e}, e_1) - \bar{\lambda} H(e_1, e_1) = 0$
set $\bar{\lambda} = \dfrac{H(\tilde{e}, e_1)}{H(e_1, e_1)}$

What if $H(e_1, e_1) = 0$?

Prop: Given a positive definite Hermitian form H, and given any flag $V_\bullet$, $\exists$ an orthogonal basis $\{e_i\}$ adapted to $V_\bullet$.

A Hermitian form H on $\mathbb{C}^n$ is equivalent to one of the following kind

$$H(z) = \sum_{i=1}^{p} |z_i|^2 - \sum_{j=1}^{q} |z_{p+j}|^2$$

- then positive definite $\iff H = |z_1|^2 + \cdots + |z_n|^2$ $(p = n)$
- If $p+q = n$, $p > 0$, $q > 0$

Ex $\mathbb{C}^2$ $\quad H(z) = |z_1|^2 - |z_2|^2$

then there are $z = (z_1, z_2) \neq 0$ s.t. $H(z) = 0$

e.g. $z = (1, 1)$ $\quad z = (1, i)$

Q: $Null(H) = \{z \in \mathbb{C}^2 \mid |z_1|^2 - |z_2|^2 = 0\}$

is it a vector space?

$(1,1) \in Null(H)$ $\quad (1,-1) \in Null(H)$ $\quad (2,0) \notin Null(H)$

Let H be a $n \times n$ hermitian matrix

$$H = \begin{bmatrix} H_{11} & H_{12} & & \vdots \\ H_{21} & H_{22} & & \vdots \\ \vdots & & & H_{nn} \end{bmatrix} \quad [H_{11}]\ 1\times 1 \quad \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\ 2\times 2$$

$\Delta_1 = \det(H_{11})$ $\quad \Delta_2 = \det(\ldots)$

? only to H

principal minors:

$$\Delta_k = \det\left(\begin{matrix}\text{upper left } k\times k \\ \text{submatrix of } H\end{matrix}\right)$$

Recall: if H is a Hermitian matrix, i.e. $H = \overline{H}^t$

then $\det H = \det(\overline{H}^t) = \overline{\det(H^t)} = \overline{\det(H)}$ (i.e., $\det(H) \in \mathbb{R}$)

Assume that $\Delta_k \neq 0$ for $k = 1, 2, \ldots, n$

Thm: Let q be the number of sign changes in $(\Delta_0, \Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n)$ then q = the negative inertia index of H

Ex: if H is a diagonal matrix

$$H = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$ negative index $= 2$ $\quad \Delta_0 = 1, \Delta_1 = 1, \Delta_2 = -1, \Delta_3 = -1, \Delta_4 = 1$

$H = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\Delta_k = \lambda_1 \cdots \lambda_k$

if $\lambda_k < 0$, then Δ_{k-1} and Δ_k has a sign change

- Let $V = \mathbb{C}^n$ $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$

Let $V_\bullet$ be the standard flag for $\{e_i\}$

- By Gram Schmidt process obtain a new basis $f_1, f_2, \dots, f_n$ that $\mathrm{span}(f_i) = \mathrm{span}(e_i) = V$

is ~~orthogonal~~

and $H(f_i, f_j) = 0$ for $i \neq j$ $\quad H(\lambda v) = |\lambda|^2 H(v)$

$H(f_i, f_i) = \pm 1$

$D = C^* H C$ $\quad C = \begin{pmatrix} & * \\ 0 & \end{pmatrix}$ $\quad \Delta_k(D)$ and $\Delta_k(H)$ has the same sign

Let D_k be the upper left $k \times k$ submatrix of D

$H_k \dots \quad C_k$

then $D_k = C_k^\dagger H_k C_k$

$\det(D_k) = \det(C_k^\dagger) \cdot \det(H_k) \cdot \det(C_k)$

$= \overline{\det(C_k)} \cdot \det(C_k) \cdot \det(H_k)$

$= |\det(C_k)|^2 \cdot \det(H_k)$

- Eigenvalue Problem:

V: v.s./$\mathbb{C}$; $H: V \times V \to \mathbb{C}$; $\|v\|^2 := H(v, v)$

Claim: $\|v + w\| \leq \|v\| + \|w\|$

General Definition: $\quad T \in \mathrm{End}(V)$ Endomorphism

Let V be a vector space $/\mathbb{K}$ Let $T: V \to V$ be a linear map

we say $v \in V$ is an eigenvector of T, with eigenvalue $\lambda \in \mathbb{K}$, if $T(v) = \lambda \cdot v$

Dream Case:

Given $T: V \to V$, $\exists$ a basis of V, $\{e_1, \dots, e_n\}$ and numbers $\{\lambda_1, \dots, \lambda_n\}$ s.t. $T(e_i) = \lambda_i e_i$

Dream may not be true:

Eg: $V = \mathbb{R}^2$, $\mathbb{K} = \mathbb{R}$

$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad T(v)$ = rotate V by $90°$

one cannot diagonalize it $\quad$ not in $\mathbb{R} \cdot v$

Ex: $\mathbb{K} = \mathbb{C}$ one can have one eigenvector

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T(e_1) = 1 \cdot e_1$ cannot find a second one

· Diagonalize a Hermitian form $H: V\times V \to \mathbb{C}$ means finding basis $e_1, \dots, e_n$ s.t. $H_{ij} = H(e_i, e_j)$ is diagonal

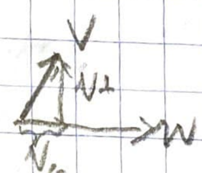
· V: n-dim$_{\mathbb{C}}$ complex v.s.
· $\langle -,- \rangle : V \times V \to \mathbb{C}$ Hermitian form
$\langle \lambda v, \mu w \rangle = \bar{\lambda} \cdot \mu \langle v, w \rangle$, $\lambda, \mu \in \mathbb{C}$ $v, w \in V$
positive definite $v \neq 0 \Leftrightarrow \langle v, v \rangle > 0$
· $|v| = \sqrt{\langle v, v \rangle}$ "length / norm of v"

Cauchy-Schwarz ineq:

Given any $v, w \in V$, we have
$|\langle v, w \rangle|^2 \leq |v|^2 \cdot |w|^2$

Proof: if $|v| = 0$, or $|w| = 0$, then $\langle v, w \rangle = 0$, then nothing to prove
· if $|v|, |w| \neq 0$, then we can define $\hat{v} = v/|v|$ $\hat{w} = w/|w|$
then $|\langle v, w \rangle|^2 \leq |v|^2 \cdot |w|^2$
$\Leftrightarrow \left| \frac{\langle v, w \rangle}{|v| \cdot |w|} \right|^2 \leq 1 \Leftrightarrow \left| \langle \frac{v}{|v|}, \frac{w}{|w|} \rangle \right|^2 \leq 1 \Leftrightarrow |\langle \hat{v}, \hat{w} \rangle| \leq 1$

Simplify notation $\hat{v} \rightsquigarrow v$ $\hat{w} \rightsquigarrow w$ $|v| = 1$, $|w| = 1$



$v = v_{\parallel} + v_{\perp}$
$v_{\parallel} = \langle w, v \rangle \cdot w$
$v_{\perp} = v - \langle w, v \rangle w$

indeed $\langle w, v_{\perp} \rangle$
$= \langle w, v - \langle w, v \rangle w \rangle$
$= \langle w, v \rangle - \langle w, v \rangle \langle w, w \rangle$
$= \langle w, v \rangle - \langle w, v \rangle = 0$

$0 \leq \langle v_{\perp}, v_{\perp} \rangle = \langle v_{\perp}, v - \lambda w \rangle = \langle v_{\perp}, v \rangle - \lambda \langle v_{\perp}, w \rangle$
$= \langle v - \lambda w, v \rangle = \langle v, v \rangle - \langle \lambda w, v \rangle$
$= 1 - \bar{\lambda} \cdot \langle w, v \rangle$
$= 1 - \bar{\lambda} \cdot \lambda = 1 - |\lambda|^2$
$\Rightarrow |\lambda|^2 \leq 1 \Rightarrow |\langle w, v \rangle|^2 \leq 1$

Triangle ineq.

$|v + w| \leq |v| + |w| \Leftrightarrow |v + w|^2 \leq (|v| + |w|)^2$
$\Leftrightarrow |v|^2 + |w|^2 + \langle v, w \rangle + \langle w, v \rangle \leq |v|^2 + |w|^2 + 2|v||w|$
$\Leftrightarrow Re(\langle v, w \rangle) \leq |v| \cdot |w|$ $\Leftarrow |\langle v, w \rangle| \leq |v| \cdot |w|$

Let $V, \langle -,- \rangle$ be a Hermitian v.s. V: v.s./$\mathbb{C}$

$\Rightarrow \exists$ ONB $\{e_1, \dots, e_n\}$

$\Rightarrow \exists$ isomorphism between

$$(V, \langle , \rangle) \xrightarrow{\sim} (\mathbb{C}^n, \langle -,- \rangle_{std})$$

$v = v_1 e_1 + \dots + v_n e_n \mapsto (v_1, \dots, v_n)$ $\quad \langle z, w \rangle_{std} = \sum_{i=1}^{n} \bar{z}_i w_i$

$|v|^2 = \sum_{i=1}^{n} |v_i|^2$ $\quad z = (z_1, \dots, z_n)$ $w = (w_1, \dots, w_n)$

Baby Example of identification of V with V^*

Let V be a v.s. over $\mathbb{R}$

e.g. $V = \mathbb{R}^2$, $\mathbb{R}^2$ with std $\langle -,- \rangle$ Euclidean product

For each $v \in V$, we can define an element $\phi_v \in V^*$ as

$\phi_v(w) = \langle v, w \rangle$ $\quad$ $\mathbb{R}$-vector space

This gives a map of set $V \Rightarrow V^*$, $v \longmapsto \phi_v$

Similarly, for V, a Hermitian v.s.,

we have $\Phi: V \longrightarrow V^*$

a map of sets $\quad v \longmapsto \phi_v = \langle v, - \rangle$

$\psi_v = \langle -, v \rangle$

$\psi_v : V \longrightarrow \mathbb{C}$

$w \longmapsto \langle w, v \rangle$

this ψ isn't a $\mathbb{C}$-linear map

$\psi_v(\lambda \cdot w) = \bar{\lambda} \cdot \psi_v(w)$

Φ is not $\mathbb{C}$-linear but it's anti-$\mathbb{C}$-linear $\quad \Phi(v) = \phi_v$ $\quad \Phi(i v) = (-i)\phi_v$

$\Phi(\lambda v) = \bar{\lambda} \cdot \phi_v$

Adjoint: Give a linear map of Hermitian v.s.

$A: V \to W \Rightarrow A^*: W^* \longrightarrow V^*$ (adjoint map of dual space)

$\Rightarrow$

$$\begin{tikzcd} W^* \arrow[r] & V^* \arrow[d, "\Phi_V^{-1}"] \\ W \arrow[u, "\Phi_W", "\sim"'] \arrow[r, "A^*"] & V \end{tikzcd}$$

· Given any $v \in V$ $w \in W$ $\quad \boxed{\text{change notation } A^* = A^\dagger}$

$\langle w, Av \rangle_W = \langle A^* w, v \rangle_V$

this is the defining property of A^*

Hermitian v.s.

Concretely: if $A: \mathbb{C}^n \longrightarrow \mathbb{C}^m$ with matrix A_{ij}

then $A^*: \mathbb{C}^m \longrightarrow \mathbb{C}^n$ has matrix $(A^*)_{ij} = \overline{A_{ji}}$

Normal Operator

Let $(V, \langle \cdot, \cdot \rangle)$ be a Hermitian v.s.

$A: V \longrightarrow V$ $\mathbb{C}$ linear map $\quad A^*: V \longleftarrow V$

we say A is normal if A^* commute with A $(A^* \cdot A = A \cdot A^*)$

Ex 1: • if $A^* = A$, $\leftarrow$ we say A is self-adjoint
then A is normal

• if $A^* = A^{-1}$ $\leftarrow$ we say A is a unitary operator
then A is normal

• if $A^* = -A$, then A is skew-adjoint. A is normal

Ex: $A: \mathbb{C}^2 \to \mathbb{C}^2$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$A^*A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad AA^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Spectral Thm for Normal Operator

Let $A: V \longrightarrow V$ linear map of Hermitian v.s.

Then A is normal $\Leftrightarrow$ $\exists$ ONB $\{e_1, \cdots, e_n\}$ s.t. $Ae_i = \lambda_i e_i$

Proof: (sketch):

$\Leftarrow$ A is represented by a diagonal matrix

$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, $A^* = \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix}$, $AA^* = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix}$

$\Rightarrow$ ① find one eigenvalue $\lambda \in \mathbb{C}$ (a soln to $\det(A - \lambda I) = 0$)

② eigenspace

$W_\lambda = \{v \mid Av = \lambda v\} = \ker(A - \lambda I)$

invariant under multiplication by A^*

③ $W_\lambda^\perp$ is also invariant under A & A^*

suppose the problem is solved $V' = W_\lambda^\perp$, $A' = A|_{W_\lambda^\perp}$