Concretely: if $A: \mathbb{C}^n \longrightarrow \mathbb{C}^m$ hermitian v.s.

with matrix A_{ij}

then $A^*: \mathbb{C}^m \longrightarrow \mathbb{C}^n$ has matrix $(A^*)_{ij} = \overline{A_{ji}}$

Normal Operator

Let $(V, \langle \cdot, \cdot \rangle)$ be a Hermitian v.s.

$A: V \longrightarrow V$ $\mathbb{C}$ linear map $\quad A^*: V \longleftarrow V$

we say A is normal if A^* commute with A $(A^* A = A A^*)$

Ex 1: • if $A^* = A$, ← we say A is self-adjoint

then A is normal

• if $A^* = A^{-1}$ ← we say A is a unitary operator

then A is normal

• if $A^* = -A$, then A is skew-adjoint. A is normal

Ex: $A: \mathbb{C}^2 \to \mathbb{C}^2$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$A^*A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad AA^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Spectral Thm for Normal Operator

Let $A: V \to V$ linear map of Hermitian v.s.

Then A is normal $\Leftrightarrow \exists$ ONB $\{e_1, \dots, e_n\}$ s.t. $Ae_i = \lambda_i e_i$

Proof (sketch): $\Leftarrow$ A is represented by a diagonal matrix

$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $A^* = \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{pmatrix}$ $AA^* = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix}$

$\Rightarrow$ ① find one eigenvalue $\lambda \in \mathbb{C}$ (a soln to $\det(A - \lambda I) = 0$)

② eigenspace

$W_\lambda = \{v \mid Av = \lambda v\} = \ker(A - \lambda I)$

invariant under multiplication by A^*

③ $W_\lambda^\perp$ is also invariant under A, & A^*

suppose the problem is solved $V' = W_\lambda^\perp$, $A' = A|_{W_\lambda^\perp}$

Abstract Version:

- V: complex v.s. $/\mathbb{C}$ of dim n
- $\langle -,- \rangle$ pos. def. Hermitian form on V
- $A: V \to V$ $\mathbb{C}$-linear operator

$A^*: V \to V$ adjoint

A^* is defined s.t. $\forall v_1, v_2 \in V$ $\langle v_1, Av_2 \rangle = \langle A^* v_1, v_2 \rangle$

we say A is normal, if $[A, A^*] = 0$ $\quad A, B: V \to V$ $[A \circ B - B \circ A] =: [A,B]$

Concrete Version:

- $V = \mathbb{C}^n$, $\langle -,- \rangle : V \times V \to \mathbb{C}$ $\langle (z_1, \cdots, z_n), (w_1, \cdots, w_n) \rangle$

$= \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n$

- A: $n \times n$ matrix with entries in $\mathbb{C}$

given $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n$ apply A to z, to get Az

- A^* is the hermitian conjugate of A $\quad A^* = \bar{A}^T = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{21} \\ \bar{A}_{12} & \bar{A}_{22} \end{bmatrix}$

A is normal if $AA^* = A^*A$

Thm (Spectral Thm for Normal Operators)

A is normal $\iff$ $\exists$ an ONB of V consist of eigenvectors for A

Thm (Concrete Version)

If A is an $n \times n$ matrix s.t. $[A, A^*] = 0$, then $\exists$ a unitary matrix U s.t. U^*AU is diagonal

Two Versions are equivalent (from abstract to concrete):

- Find an arbitrary ONB for V, $\{e_1, \cdots, e_n\}$

(has nothing to do with A)

(e_i) allows one to identify $\Phi: V \to \mathbb{C}^n$ preserving $\langle -,- \rangle$

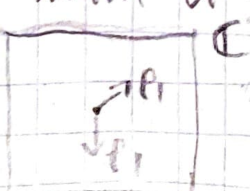
$\langle v_1, v_2 \rangle_V = \langle \Phi(v_1), \Phi(v_2) \rangle_{\mathbb{C}^n}$

Let $\{f_1, \cdots, f_n\}$ be another ONB for V s.t. $A(f_i) = \lambda_i f_i$ $\lambda_i \in \mathbb{C}$

then $(e_1, \cdots, e_n) = (f_1, \cdots, f_n) \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots \\ u_{21} & & \\ & & \\ & & u_{nn} \end{pmatrix}}_{U} \; (U^* = U^{-1})$

Lemma: if $(e_i)_{i=1}^n$, $(f_i)_{i=1}^n$ are ONB for V, then $(e_1, \cdots, e_n) = (f_1, \cdots, f_n) U$ with $U^* = U^{-1}$

Ex: $V = \mathbb{C}$



$|e_1| = 1$ $\quad e_1 = f_1 \cdot e^{i\theta}$ $\quad U = e^{i\theta}$ $\quad U^* = e^{-i\theta}$

$|f_1| = 1$

$U \cdot U^* = 1 \; (i.e. \; U^* = U^{-1})$

Suppose $e_1 = e^{i\theta_1}$, $f_1 = e^{i\theta_2}$

then $\langle e_1, f_1 \rangle = e^{-i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_2 - \theta_1)}$ $\qquad v = \langle f_1, v \rangle f_1$

$\langle f_1, e_1 \rangle = e^{i(\theta_1 - \theta_2)}$ $\qquad \Uparrow$

$e_1 = f_1 \cdot e^{i\theta} \Leftrightarrow e^{i\theta_1} = e^{i\theta_2} \cdot e^{i(\theta_1 - \theta_2)}$

$e^{i\theta} = e^{i(\theta_1 - \theta_2)} = \langle f_1, e_1 \rangle$ $\qquad e_1 = \langle f_1, e_1 \rangle f_1$

If (e_i) is an ONB of V, then $\forall v \in V$ $v = \langle e_1, v \rangle e_1 + \cdots + \langle e_n, v \rangle e_n$

$e_i = \sum_{j=1}^{n} \langle f_j, e_i \rangle f_j$

$$(e_1, \cdots, e_n) = (f_1, \cdots, f_n) \begin{pmatrix} \langle f_1, e_1 \rangle & & \cdots \\ \langle f_2, e_1 \rangle & & \vdots \\ \vdots & & \vdots \\ \langle f_n, e_1 \rangle & & \langle f_n, e_n \rangle \end{pmatrix} \qquad U_{ij} = \langle f_i, e_j \rangle$$

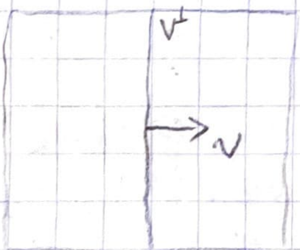
$(U^*)_{ij} = \overline{U_{ji}} = \overline{\langle f_j, e_i \rangle} = \langle e_i, f_j \rangle$

$(U \cdot U^*)_{ij} = \sum_{k=1}^{n} U_{ik} (U^*)_{kj} = \sum_{k=1}^{n} \langle f_i, e_k \rangle \langle e_k, f_j \rangle = \langle f_i, f_j \rangle$

$\langle w, v \rangle = \langle w, \sum_k \langle e_k, v \rangle e_k \rangle = \sum_k \langle e_k, v \rangle \langle w, e_k \rangle$

$= \sum_k \langle w, e_k \rangle \langle e_k, v \rangle$

Proof of Thm:



V

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A' \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] & A \\ A' \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] \arrow[r, dotted, "?"] & A' \end{tikzcd}$$

Step 1: want to find an eigenvector (actually, an eigenspace)

if $A \cdot v = \lambda v$ then $(A - \lambda I) \cdot v = 0$ i.e. $v \in \ker(A - \lambda I)$

$\Rightarrow \det(A - \lambda I) = 0$

solve $\det(A - \lambda I) = 0$

$\quad \|$

degree n polynomial in λ

$\because$ we are working in $\mathbb{C}$

$\therefore$ by fundamental theorem of algebra

$\exists$ a factorization $\det(A - \lambda I) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$

pick one sol'n to $\det(A-\lambda I)=0$

say $\lambda=\lambda_1$

then let $W_\lambda = \ker(A-\lambda I) = \{v\in V \mid (A-\lambda I)v=0\}$

Let $W_\lambda^\perp = \{w\in V \mid \langle w, v\rangle = 0, \forall v\in W\}$

② A and A^* preserve W_λ

$\forall v\in W_\lambda$, we need to show $A\cdot v\in W_\lambda$, $A^*v\in W_\lambda$

$$Av\in W_\lambda \iff A\cdot(Av)=\lambda\cdot(Av)$$
$$\iff A(\lambda v) = \lambda\cdot Av \checkmark$$

$$A^*v\in W_\lambda \iff A(A^*v)=\lambda(A^*v)$$
$$\iff A^*Av = \lambda A^*v \iff A^*(\lambda v)=\lambda A^*v \checkmark$$

③ A and A^* preserves $W_\lambda^\perp$

Suppose $w\in W_\lambda^\perp$, then $Aw\in W_\lambda^\perp$, and $A^*w\in W_\lambda^\perp$

$$Aw\in W_\lambda^\perp \iff \forall v\in W_\lambda \ \langle v, Aw\rangle=0$$
$$\iff \forall v\in W_\lambda \ \langle A^*v, w\rangle = 0$$

this true since $A^*v\in W_\lambda$, $w\in W_\lambda^\perp$

$$A^*w\in W^\perp \iff \forall v\in W_\lambda, \ \langle v, A^*w\rangle=0$$
$$\iff \forall v\in W_\lambda, \ \langle Av, w\rangle = 0$$

$\because Av\in W_\lambda$, $w\in W_\lambda^\perp$ $\therefore$ this holds

From $\mathbb{R}$-v.s. to $\mathbb{C}$-v.s.

$\mathbb{R} \rightsquigarrow \mathbb{C}$

$\mathbb{R}^n \rightsquigarrow \mathbb{C}^n$

Let $V_\mathbb{R}$ be a real vector space of $\dim_\mathbb{R} n$

we define $V_\mathbb{C} = \{x+iy \mid x\in V_\mathbb{R}, y\in V_\mathbb{R}\} \overset{\text{set}}{=} V_\mathbb{R}\times V_\mathbb{R}$

let $z=a+ib$, $a,b\in\mathbb{R}$

then $v=x+iy\in V_\mathbb{C}$

$$z\cdot v = (a+ib)(x+iy) = (ax-by)+i(bx+ay)$$

inner product space

$\mathbb{R}$-vector space

- $V=\mathbb{R}^n$
- $\langle , \rangle$: positive definite $\langle v,v\rangle>0 \ \forall v\neq 0$
  symmetric bilinear form on V

$A: V\to W$ $\quad A^*: W\to V$ adjoint map

$$\begin{tikzcd} W^* \arrow[r, "A^*"] & V^* \\ W \arrow[u, "\cong"] \arrow[r, dashed] & V \arrow[u, "\cong"'] \end{tikzcd}$$

Concretely, given ONB of v, w

$A^* = A^t$

Complexify : $V_{\mathbb{C}}$

: $\langle -, - \rangle_{V_{\mathbb{C}}}$, choosing ONB of $V_{\mathbb{R}}$ as ONB of $V_{\mathbb{C}}$

$$\mathbb{R} \hookrightarrow \mathbb{C}$$
$$V_{\mathbb{R}} \hookrightarrow V_{\mathbb{C}}$$
$$x \mapsto (x + i0)$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, hook] & V_{\mathbb{R}} \arrow[l, "\sim"'] \arrow[d, hook] \\ \mathbb{C}^n & V_{\mathbb{C}} \arrow[l] \end{tikzcd}$$

in one word: complexification of $V_{\mathbb{R}}$ with basis $(e_1, \cdots, e_n)$ is to allow
$\mathbb{C}$ - coefficient, $c_1 e_1 + \cdots + c_n e_n$ $c_i \in \mathbb{C}$

Thm: Let $(V_{\mathbb{R}}, \langle -, - \rangle)$ be a real, vector space
Let $(V_{\mathbb{C}}, \langle -, - \rangle)$ be its complexification
Suppose $A_{\mathbb{R}} : V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a normal operator
$[A_{\mathbb{R}}, A_{\mathbb{R}}^*] = 0$
Then, its complexification $A_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ has eigenvectors in $V_{\mathbb{C}}$
i.e., $\exists$ eigenvectors of A $\{e_1, \cdots, e_n\} \subset V_{\mathbb{C}}$ that forms an ONB of $V_{\mathbb{C}}$

If eigenvalue $\lambda_i \in \mathbb{R}$, then $\overline{\lambda_i}$ is also an eigenvalue and if
e_i is eigen vector $A e_i = \lambda_i e_i$
then $\overline{e_i}$ is e.v. $A \overline{e_i} = \overline{\lambda_i} \overline{e_i}$

Differential Equation:

unknown function: $f(x)$ or $x(t)$

① If it's a function of one variable then ODE
Else it's PDE (partial differential eq.)

Linear ODE: equation only involves $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$
not $x^2(t)$, $x^3(t)$, or $\dot{x}\,x^2(t)$

Non-linear ODE ex: $\frac{dx(t)}{dt} = x(t)^2$

Ex: (1) $\frac{dx}{dt} = 0$ sol'n $x(t) = \text{constant}$

(2) $\frac{dx}{dt} = C_1$ sol'n $x(t) = C_0 + C_1 \cdot t$

(3) $\frac{dx}{dt} = f(t)$ — $f(t)$ is a "nice" function

$x(t) = \text{constant} + \int^t f(t)\,dt = x(t_0) + \int_{t_0}^{t} f(s)\,ds$

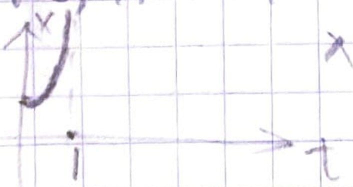
(4) $\dot{x}(t) = x(t)$ sol'n: $x(t) = C \cdot e^t = x(t_0)\, e^{t-t_0}$

$\frac{dx}{dt} = x \iff \frac{dx}{x} = dt \iff d(\ln x) = dt \iff \ln x = t + c$
$\iff x = e^{t+c_0} = e^{c_0} \cdot e^t$

(5) $\dot{x}(t) = x^2(t)$ $\quad \frac{dx}{dt} = x^2 \iff \frac{dx}{x^2} = dt \iff d(-\frac{1}{x}) = dt$
$\iff -\frac{1}{x} = c + t \iff x = \frac{1}{-c-t}$

suppose $t=0$, $x=1$ $\quad 1 = \frac{1}{-c} \iff c = -1$ $\quad = \frac{1}{(-c)-t}$

$x = \frac{1}{1-t}$



(6) $\frac{dx}{dt} = x + t$ $\quad x(t) = u(t)\, e^t$ — new unknown function

$\frac{d(u(t) \cdot e^t)}{dt} = u(t) \cdot e^t + t$

$\dot{u} e^t + u e^t = u e^t + t$

$\dot{u} e^t = t$

$\dot{u} = t \cdot e^{-t}$

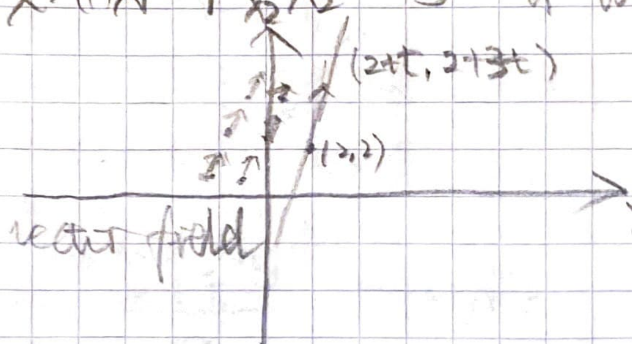
$u(t) = u(t_0) + \int_{t_0}^{t} s \cdot e^{-s}\,ds = C + \int t e^{-t}\,dt$

unknown function: $x_1(t)$, $x_2(t)$

Eq: $\dot{x}_1 = F_1(x_1, x_2, t)$ $\quad \dot{x}_2 = F_2(x_1, x_2, t)$

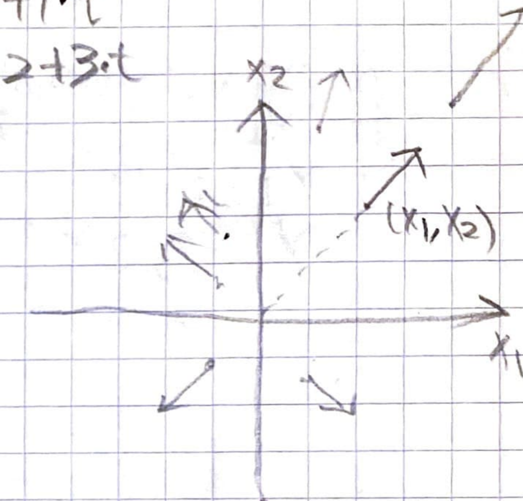
Ex: (1) $\dot{x}_1 = 1$, $\dot{x}_2 = 3$ if we have boundary (initial) condition

$t=0 \begin{cases} x_1 = 2 \\ x_2 = 2 \end{cases}$



$x_1 = 2 + 1 \cdot t$

$x_2 = 2 + 3 \cdot t$



(2) $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2 \end{cases}$ $\quad x_1(t) = C_1 e^t = x_1(0) e^t$

$x_2(t) = C_2 e^t = x_2(0) e^t$

$\dot{x}_1 = -x_2$

$\dot{x}_2 = x_1$

$x_1 = \cos t$ $\quad \dot{x}_1 = -\sin(t) = -x_2$

$x_2 = \sin t$ $\quad \dot{x}_2 = \cos(t) = x_1$

another sol'n $\begin{cases} x_1 = -\sin(t) \\ x_2 = \cos(t) \end{cases}$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = \text{soln}_1 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = \text{soln}_2$$

Claim General solution = $C_1 \,\text{soln}_1 + C_2 \,\text{soln}_2$

Eq: $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow \left[\frac{d}{dt} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$

$\Leftrightarrow \begin{pmatrix} \frac{d}{dt} & 1 \\ -1 & \frac{d}{dt} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$

(1) $\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\rightsquigarrow$ sol'n radial motion outward

$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$\begin{cases} x_1 = C_1 e^{\lambda_1 t} \\ x_2 = C_2 e^{\lambda_2 t} \end{cases}$

$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$

the unknown function only depends on t

Linear ODE

Last time:

- unknown function: $x(t)$
- constraint equation: $\frac{d}{dt}x(t) = a\cdot x + b$

Linear means only linear term & const. term appears (no x^2)

Special Ex.

$$\frac{d}{dt}x(t) = x(t) \rightsquigarrow x(t) = c\cdot e^t$$

$$\frac{dx}{dt} = 5x \qquad x(t) = c\cdot e^{5t} \qquad \frac{dx}{dt} = \lambda\cdot x \quad (\lambda\in\mathbb{C}) \quad x(t) = ce^{\lambda t}$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ & \lambda_2 & & 0 \\ & & \ddots & \vdots \\ 0 & & & \lambda_n\end{pmatrix}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix} \iff \begin{cases} \frac{d}{dt}x_1(t) = \lambda_1\cdot x_1(t) \\ \vdots \\ \frac{d}{dt}x_n(t) = \lambda_n\cdot x_n(t)\end{cases}$$

$$\Rightarrow x_k(t) = C_k\cdot e^{\lambda_k\cdot t}$$

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t)\end{pmatrix} = \begin{pmatrix} & & \\ & \underset{\text{invertible matrix}}{\underset{\uparrow}{C}} & \\ & & \end{pmatrix}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_n\end{pmatrix} = \frac{d}{dt}\left(C\cdot\vec{x}(t)\right)$$

$$= C\cdot\frac{d}{dt}\vec{x}(t)$$

$$= C\cdot\Lambda\cdot\vec{x}(t)$$

$$= (C\cdot\Lambda\cdot C^{-1})y(t)$$

$$\frac{d}{dt}\vec{x} = \Lambda\vec{x}$$

need to start from:

$\frac{d}{dt}y = A\cdot y$ Try to diagonalize A as

$A = C\Lambda C^{-1}$ (Λ ← diagonal) Then we have $x(t) = C^{-1}y$, $\frac{d}{dt}x = Ax$

$$\Rightarrow x(t) = \begin{pmatrix} c_1e^{\lambda_1 t} \\ \vdots \\ c_ne^{\lambda_n t}\end{pmatrix}$$

$$y(t) = \begin{pmatrix} C \end{pmatrix}\begin{pmatrix} c_1e^{\lambda_1 t} \\ \vdots \\ c_ne^{\lambda_n t}\end{pmatrix}$$

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ $\det(A-\lambda I) = 0 \rightarrow \lambda = \pm i$

for $\lambda = i$, we need to solve

$$\begin{pmatrix} -i & -1 \\ 1 & -i\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = 0, \text{ one sol'n is } \begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} i \\ 1\end{pmatrix} = v_1$$

for $\lambda = -i$ $(A-\lambda I)v = 0$, one sol'n is $\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} -i \\ 1\end{pmatrix} = v_2$

Let $C = \begin{pmatrix} (v_1) & (v_2)\end{pmatrix}$ $AC = C\cdot\Lambda$ $A = C\Lambda C^{-1}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = \underset{C}{\begin{pmatrix} i & -i \\ 1 & 1\end{pmatrix}}\underset{\Lambda}{\begin{pmatrix} i & \\ & -i\end{pmatrix}}\underset{C^{-1}}{\begin{pmatrix} 1 & i \\ -1 & i\end{pmatrix}\cdot\frac{1}{2i}}$$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{set} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{the } \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_1 = C_1 e^{it} \quad x_2 = C_2 e^{-it}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} C_1 e^{it} \\ C_2 e^{-it} \end{pmatrix}$$

what happens if $A \neq C \Lambda C^{-1}$ cannot be diagonalized

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = 0 \end{cases}$

$\iff \ddot{x}(t) = 0$

$x_2(t) = C_2 \quad x_1(t) = C_2 \cdot t + C_1$

$F = ma = \cancel{m\left(\frac{d}{dt}\right)x} \; C = m\left(\frac{d}{dt}\right)^2 x(t) \iff \ddot{x}(t) = C$

One can always reduce an equation about higher order derivatives $\left(\frac{d}{dt}\right)$ to first order derivatives, at the cost of introducing auxillary variables $\iff$

$x_1(t) = x(t)$

$x_2(t) = \dot{x}(t)$

$\dot{x}_1 = x_2$

$\dot{x}_2 = C$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix} n \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2(t) \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

n

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = 0 \end{cases} \iff \left(\frac{d}{dt}\right)^n x_1(t) = 0$$

$x_1(t) = \frac{t^{n-1}}{(n-1)!} C_{n-1} + \cdots + t_1 C_1 + C$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Big(\lambda \cdot Id + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N}\Big)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \frac{d}{dt}\vec{x} = (\lambda + N)\vec{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \qquad \frac{d}{dt}(e^{\lambda t} \cdot u) = (\lambda + N)(e^{\lambda t} \cdot u)$$

$\vec{x} = e^{\lambda t} \vec{u}(t)$

$\lambda e^{\lambda t} u + e^{\lambda t} \cdot \dot{u} = \lambda e^{\lambda t} u + N e^{\lambda t} u$

$e^{\lambda t}\dot{u} = e^{\lambda t} \cdot Nu \iff \dot{u} = Nu$

In general, if $N_n = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$ then $\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\lambda + N_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

can be solved $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = e^{\lambda t}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $\quad \frac{d}{dt}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

Jordan Decomposition Thm / $\mathbb{C}$

- say A is an $n \times n$ matrix, then $\exists$ an invertible matrix C s.t. $A = C \begin{pmatrix} \boxed{\lambda_1 + N_{n_1}} & & & \\ & \boxed{\lambda_2 + N_{n_2}} & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} C^{-1}$

blocks λ_i can equal to λ_j

$N_1 = (0)$

$N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\vdots$

$N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

- step 1: how to get $(\lambda_1 \cdots \lambda_r)$
  solve eq. $\det(A - \lambda I) = 0$

say we have some decomposition $\det(A - \lambda I)$

$\det(A - \lambda I) = C(\lambda - a_1)^{m_1} \cdots (\lambda - a_k)^{m_k}$

for each root a_i, we can define the generalized eigenspace

$W_{a_i} = \ker((A - a_i)^{m_i})$

Ex: $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 3I + N_3$ $\quad \det(A - \lambda) = \det\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3$

nilpotent matrix

$A - 3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad N_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad N_3^3 = 0$

- $\ker(N_3) = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{C}^3 \;\middle|\; N_3 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\ker(N_3^2) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ $\quad \ker(N_3^3) = \mathbb{C}^3$

Ex $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \boxed{3+N_2} & \\ & \boxed{3} \end{pmatrix}$

$(A-3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad (A-3)^2 = 0$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix} = 0 \iff v_2 = 0$

$\ker(A-3) = \mathrm{span}(e_1, e_3)$ $\quad \ker((A-3)^2) = \mathbb{C}^3$

A : $n \times n$ matrix / $\mathbb{C}$

$\det(A - \lambda I) = C(\lambda - a_1)^{m_1} \cdots (\lambda - a_k)^{m_k}$

$W_{a_i} = \ker((A - a_i)^{m_i})$

Now $A\big|_{W_{a_i}}$ has only eigenvalue a_i

$(A - a_i)\big|_{W_{a_i}}$ is nilpotent, i.e. $\left((A - a_i)\big|_{W_{a_i}}\right)^N = 0$

Ex of Nilpotent matrix of size n

$N = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$ regular nilpotent

if $N^1 \neq 0, N^2 \neq 0, \cdots, N^{n-1} \neq 0, N^n = 0$

$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 0 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$

N nilp $\iff C \cdot N \cdot C^{-1}$ nilp

$N^k = 0 \iff (CNC^{-1})^k = 0$