HW 2.

1.3.1

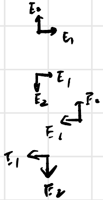
(b)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow$.

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Leftrightarrow$ mirror along $y=0$.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow$ mirror along $x=0$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ rotate. 180°

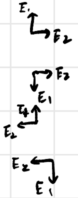


$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ mirror along $y=x$

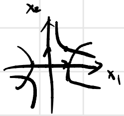
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ rotate ↶ 270°

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotate counterclockwise 90°.

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ mirror along $y=-x$

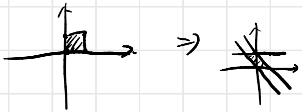


(c)



$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$

(d)



Solution 1: $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ Solution 2: composition of $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

(g)

$\because \|C\vec{x}\|^2 = \|\vec{x}\|^2$

$\therefore C\vec{x} \cdot C\vec{x} = \vec{x} \cdot \vec{x}$

$\therefore x^T C^T \cdot C\vec{x} = \vec{x} \cdot \vec{x}$

$\therefore \vec{x} \cdot \vec{x} = \vec{x} \cdot \vec{x}$

$\therefore$ orthognal

1.3.2

(a) trivial

(b)

$DA = \begin{bmatrix} a_{11}d_1 & a_{12}d_1 \\ a_{21}d_2 & a_{22}d_2 \end{bmatrix}$

$AD = \begin{bmatrix} a_{11}d_1 & a_{12}d_2 \\ a_{21}d_1 & a_{22}d_2 \end{bmatrix}$

(c).

$T_\theta R_0 = R_\theta$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

(d)

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

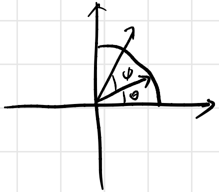
$$= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1+\sqrt{3}}{4} \\ 1-\frac{\sqrt{3}}{2} \end{pmatrix}$$

(e) $T_\phi T_\psi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi \\ \sin\phi\cos\psi + \cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi \end{pmatrix}$$

$$= T(\phi+\psi)$$



T

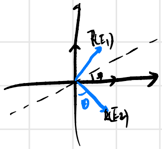
$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\psi \end{pmatrix} \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi \\ \sin\phi\cos\psi + \sin\psi\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi+\psi) \\ \sin(\phi+\psi) \end{pmatrix}$$

(f) $R_\phi R_\psi$

$$= \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}$$



$$= \begin{pmatrix} \cos\phi\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi \\ \sin\phi\cos\psi - \sin\psi\cos\phi & \sin\phi\sin\psi + \cos\phi\cos\psi \end{pmatrix}$$

$$T_{\phi-\psi} = \begin{pmatrix} \cos(\phi-\psi) & -\sin(\phi-\psi) \\ \sin(\phi-\psi) & \cos(\phi-\psi) \end{pmatrix}$$

✓

HW3

47. Show that z^{-1} is real proportional to $\bar{z}$ and find the proportionality coefficient. ✓

1.

$z = a+bi \qquad \bar{z} = a-bi$

$z^{-1} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{1}{a^2+b^2}\bar{z}$

$k = \frac{1}{a^2+b^2}$

$\because a, b \in R, \ a^2+b^2 \in R$ and $a^2+b^2 \neq 0$

$\therefore$ ✓

2. $z = re^{i\theta}$

$z^{-1} = \frac{1}{r} \cdot e^{-i\theta}$

$\bar{z} = r \cdot e^{-i\theta}$

$r^2 = a^2+b^2$

51.

$\left(\frac{\sqrt{3}+i}{2}\right)^{100}$

$\left(\frac{\sqrt{3}+i}{2}\right)^{2} = \cancel{\frac{2-2i}{4}} = \frac{1-i}{2}$

$\left(\frac{\sqrt{3}+i}{2}\right)^{4} = \left(\frac{1-i}{2}\right)^{2} = \frac{-2i}{4} = -\frac{i}{2}$

$\left(\frac{\sqrt{3}+i}{2}\right)^{8} = \frac{1}{4}$

$\left(\frac{\sqrt{3}+i}{2}\right)^{100} = \left(\left(\frac{\sqrt{3}+i}{2}\right)^{8}\right)^{12} \cdot \left(\frac{\sqrt{3}+i}{2}\right)^{4}$

$= \frac{1}{2^{24}} \cdot \left(-\frac{i}{2}\right) = -\frac{i}{2^{25}}$

2.

$= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{100}$

$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{100}$

$= \left(e^{i\frac{\pi}{6}}\right)^{100}$

$= e^{i\frac{50\pi}{3}}$

$= \cos\frac{50\pi}{3} + i\sin\frac{50\pi}{3}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$= \frac{-1+\sqrt{3}}{2}$

53. Express $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$ in terms of $\cos\theta_i$ and $\sin\theta_i$.

$e^{i(\theta_1+\theta_2)} = (e^{i\theta_1})(e^{i\theta_2})$

$\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$

$= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$+ i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2)$

56. Find all roots of polynomials:

$z^3+8, \quad z^3+i, \quad z^4+4z^2+4, \quad z^4-2z^2+4, \quad z^6+1.$

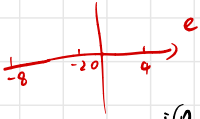
$z^3 = -8 \qquad (a+bi)^3 = -8$

$(z+2)(z^2-2z+4) = 0 \qquad a+bi = -2$

$a = -2 \ b = 0$

$z_2 = \frac{2 \pm 2\sqrt{-3}}{2}$

$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$



$e^{i\frac{\pi}{3}} \cdot 2$

$2 \cdot e^{-i\pi/3}$

$z^3+i \qquad (a+bi)^3 = -i$

$a^3 - 3ab^2 + 3a^2bi - b^3i = -i$

$r_1r_2r_3e^{i(\theta_1+\theta_2+\theta_3)} = -8$

$\begin{cases} a^3-3ab^2 = 0 & ① \\ 3a^2b-b^3 = -1 & ② \end{cases}$

if $a = 0 \quad b = 1$ ✓

$b \neq 0$

$① \Rightarrow a^2 = 3b^2$

$9b^3 - b^3 = -1$

$b^3 = -\frac{1}{8}$

$b = -\frac{1}{2}$

$a = \pm\frac{\sqrt{3}}{2}$

$z = i \ / \ z = \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i$

z^4+4z^2+4

$(z^2+2)^2=0$

$z^2+2=0$

$(a+bi)^2=-2$

$a^2-b^2-2abi=-2$

$\begin{cases} a^2-b^2=-2 \\ -2ab=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=\pm\sqrt{2} \end{cases}$

$z=\pm\sqrt{2}\,i$

z^4-2z^2+4

$a^2-2a+4=0$

44. complex number can be real & real and imeginay, but not nerther.

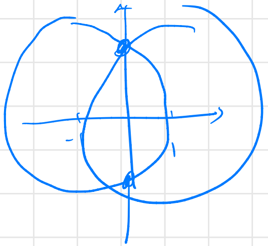
45. $\frac{1+i}{3-2i}=\frac{(1+i)(3+2i)}{13}=\frac{1+5i}{13}$

$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{-1}=\left(1\cdot e^{i\frac{\pi}{3}}\right)^{-1}=\frac{1}{e^{i\frac{\pi}{3}}}$

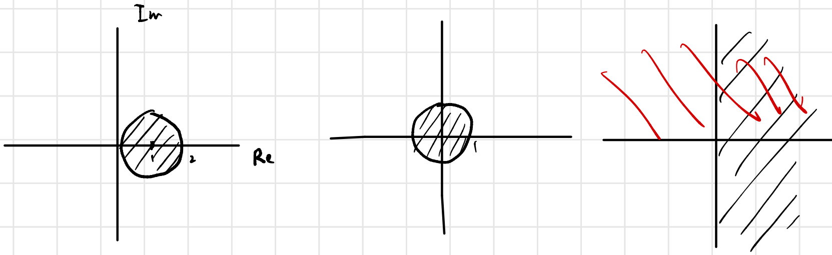
48. $|a-1+bi|=|a+1+bi|=2$

$(a-1)^2+b^2=(a+1)^2+b^2=4$

$\begin{cases} a=0 \\ b=\pm\sqrt{3} \end{cases}$



49.



$\mathrm{Re}(ai-b)\leq 0$

$-b\leq 0$

$b\geq 0$

54. $P(z_0)=a_1z_0+a_2z_0^2+\cdots+a_nz_0^n=0$

$\Updownarrow$

$P(z)=(z-z_0)Q(\cancel{z})+C$

$P(z_0)=0 \Leftrightarrow P(z)=(z-z_0)\,q(z).$

57. $p=a_nx^n+\cdots+a_0$

$(x-z_0)$ If $z_0=$ Real. $p=(x-z_0)f_1(x)$

real

$(z-z_0)=$ complex $\quad p=(z-z_0)(z-\bar{z_0})\,r_1(z)$

58.

1. $\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta$

2. $\left(\frac{1+i}{\sqrt{2}}\right)^{20} = \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}i\right)^{20}$

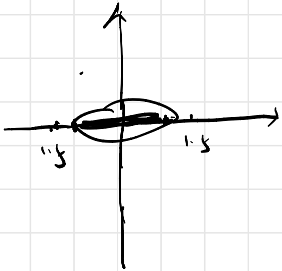
$= e^{i5\pi}$

$= e^{i\pi}$

$= \cos\pi + \sin\pi i$

$= -1$

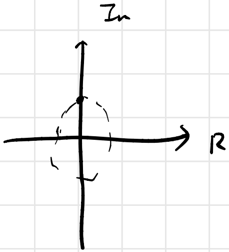
3.



$|z-1| + |z+1| \le 3$

$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{-1}$

$= e^{-i\frac{\pi}{3}}$



HW 4.

2×3   2×2   3×2.

86. ABC ×

$$BAC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}$$

BCA ×

$$CBA = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 5 \\ 7 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 17 & 4 & 31 \\ 1 & 2 & 3 \\ -7 & -2 & -13 \end{bmatrix}$$

CAB ×

ACB =

$a_{ij} = 0 \quad \forall\, i<j$

88. $c_{ij} = \sum_{k=0}^{n} a_{ik} \cdot b_{kj}.$

when $k \leq i$, $k \leq i \leq j$.

$\rightarrow a_{ik} b_{kj} = a_{ik} \cdot 0 = 0$

when $k > i$ $\quad i \leq k \leq j$.

$a_{ik} b_{kj} = 0 \cdot b_{kj} = 0.$

$\therefore c_{ij} = 0$ if $i<j$

91.

$$\text{Rotation} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdots 19$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2 \end{bmatrix} \cdots 4$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \cdots 4$$

$$= \begin{bmatrix} \cos 361^\circ & \vdots \\ - & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \cos 1^\circ & -\sin 1^\circ \\ \sin 1^\circ & \cos 1^\circ \end{bmatrix}$$

95. A and B of same n×n matrix

$$\begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{bmatrix}^2 = \begin{bmatrix} a_{11} & \cdots \\ & \end{bmatrix}^2 + 2\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots \\ & \end{bmatrix}^2$$

$$c_{ij} = \sum_{k=1}^{n} (a_{ik}+b_{ik})(a_{kj}+b_{kj})$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj} + b_{ik} \cdot b_{kj} + 2 \cdot a_{ik} \cdot b_{kj}$$

$$b_{ik} \cdot a_{kj} = a_{ik} \cdot b_{kj}$$

$$BA = AB$$

98.

$$\begin{cases} AA^{-1} = I \\ BB^{-1} = I \end{cases}$$

$$ABB^{-1}A^{-1} = I.$$

$$(AB)(B^{-1}A^{-1}) = I$$

99. $ABB^{-1}A^{-1} = I$

not necessarily $\quad (1\ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

100. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

101. $A = A^T$

$B = B^T$

$$AB = A^T B^T \neq \overset{= BA.}{B^T A^T}$$

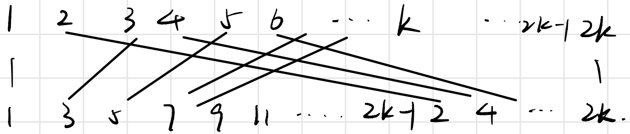
AB symmetric iff AB = BA.

Det.

111. 1, $\cos 2x$, $\cos(x+y)$

112. $a+1+1=2$ $3+3-2-6=-2.$ $1-2-1=-2$

114, $L=\frac{(\cancel{1+k})k}{2}$

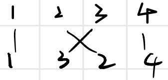


even: only cross odd numbers bigger

odd: only cross even numbers smaller than them.

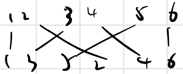
eg: $\sum_{i=1}^{n}(n-i) = (n-1)+(n-2)+\cdots 1$

$= \frac{n\cdot(n-1)}{2}$



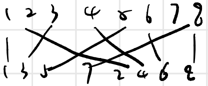
$2k=4$, $L=1$

2



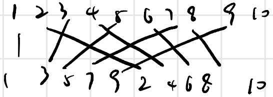
$2k=6$ $L=3$

3



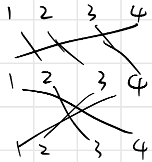
$L=6$

4



$L=10.$

116.



$\sigma=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$\sigma'=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$

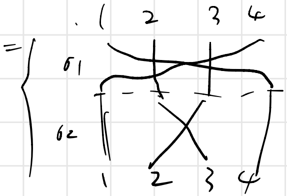
1 2 3 4

$l(\sigma^{-1})+l(\sigma)=l(w_0)=\frac{n(n-1)}{2}$

To pf: if $\sigma_2\cdot\sigma_1=w_0$

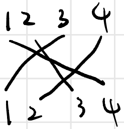
then $l(w_0)=l(\sigma_1)+l(\sigma_2)$

$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}=$ W_0.



119. $(2\ 5)(1\ 4)(1\ 3)$

Sample: ①



$L(\sigma)=5$ $\quad \varepsilon(\sigma)=-1$

②
$$\det\left(\begin{pmatrix}1&0&-1\\0&1&-1\\-1&1&0\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&a&0\\0&0&a\end{pmatrix}\right)$$

$$=\det\begin{pmatrix}1-\lambda&0&-1\\0&1-\lambda&0\\-1&1&-a\end{pmatrix}=$$

$$a_{11}a_{22}a_{13}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}$$

12)	123	123	123	123	123
123	213	132	231	321	312

$$=\lambda(1-\lambda)^2+\lambda+a-1$$
$$=a(1+\lambda^2-2\lambda)+2a-1=\lambda^3-2\lambda^2+3\lambda-1$$

③. $\forall\ L>0 \Rightarrow \exists i,\ \sigma(i)>\sigma(i+1).$

$\forall i, \sigma(i)\le\sigma(i+1),\ \exists L\le 0$

110.

$$\det(A) = \sum_{\sigma} \varepsilon(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

0

123. multiply 1st

$$100 \det(A) = \begin{pmatrix} 100 & 4 & 5 \\ 200 & 4 & 7 \\ 400 & 0 & 3 \end{pmatrix}$$

$$1000 \det(A) = \begin{pmatrix} 100 & 40 & 5 \\ 200 & 40 & 7 \\ 400 & 0 & 3 \end{pmatrix}$$

$$100 \det(A) = \begin{pmatrix} 95 & 4 & 5 \\ 249 & 4 & 7 \\ 403 & 0 & 3 \end{pmatrix}$$

$$= 13 \det(A')$$

124.

$[b_1 b_2 b_3 \quad] \; 1\times 7$

$$0 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \begin{matrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & & \cdots \\ a_3b_1 & & \\ \vdots & & a_7b_7 \end{matrix}$$

7×1

$$a_1a_2a_3 \cdots a_n \det \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_7 \\ b_1 & b_2 & \cdots & & \end{bmatrix}$$

125.

$$\begin{bmatrix} a_1, a_2, a_3, \ldots & a_n \end{bmatrix} \begin{bmatrix} a_n \cdot a_{n-1} \cdots & a_1 \end{bmatrix}$$

$$n-1+n-2+\cdots+1 = \frac{n(n-1)}{2}.$$

$$W = \binom{n}{2} = \frac{n(n-1)}{2}$$

$$n = 4k+2$$

$$\frac{2k+1 \quad 4k+1}{2}$$

127. anti-symmetric: $M^T = -M$

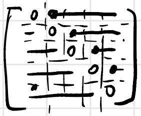
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

1. zero diagonal

$$\begin{bmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{bmatrix}$$

<u>n-1</u> option.

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$



128. $A = MBM^{-1}$

$$\det(A) = \det(MBM^{-1}) = \det(M) \cdot \det(B) \cdot \det(M^{-1})$$

$$= \det I$$

$$= \det(A).$$

141.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x \times y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

$$(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) - (x_1y_1 - x_2y_2 - x_3y_3)^2.$$

130.

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

$$\det A \, A^{-1} A = \operatorname{adj}(A) A .$$

$$\det A I = \operatorname{adj}(A) \cdot A .$$

$$\det(\det A \cdot I) = \det(\operatorname{adj}(A) \cdot A) .$$

$$= \det(\operatorname{adj}(A)) \cdot \det(A) .$$

$$A\vec{v} = 0$$

column dependent $\Leftrightarrow$ $\exists \ \vec{v} \neq 0$

$$a_1 v_1 + a_2 v_2 \cdots a_n v_n = A\vec{v} = 0 .$$

row independent $\Leftrightarrow$ $A\vec{v} = 0$ iff $\vec{v} = 0$

$$\Downarrow .$$

$$\begin{bmatrix} \\ 0\,0\,0\,0\,0 \end{bmatrix} \vec{v} = \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

131. $(-1)^{mn}$. $\det \begin{pmatrix} 0 & 1 \\ 5 & h \end{pmatrix} \det$ [illegible]

132. $1 \times (-3) + 2 \cdot (9 - 6)$

$$= 1 - 8$$

$$= 9$$

133. $1 + 27 + 8 - 6 - 6 - 6$

$$= 18 .$$

$$A^{-1} = \frac{1}{18} \begin{bmatrix} -5 & 7 & -1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$$

110. easy by def of det in permutation

118.

$L(6) = 12$ $\varepsilon(6) = 1$

$\varepsilon(6) = 1$

logarithm → algorithm consists of two transpositions

132. (a) $= 1 \cdot (1 \cdot (-3))$

$+ 2 \cdot (2 \cdot (-3))$

$= -1\cancel{5}$ 9 ✗

(b) $= 2 \times (8 - 2 - 2)$

$+ 1 \times (-1 \times 3)$

$= 5$ ✓

138. (a) $\begin{bmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 11 \end{bmatrix}$

$\det(A) = 32 + 6 + 6 + 12 + 12 - 8$

$= 60$

$\det(x_1) = \begin{bmatrix} 4 & -1 & -1 \\ 11 & 4 & -2 \\ 11 & -2 & 4 \end{bmatrix} = 64 + 22 + 22 + 44 - 16 + 44$

$= 180$

$\det(x_2) = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 11 & -2 \\ 3 & 11 & 4 \end{bmatrix} = 88 - 33 - 24 + 33 - 48 + 44$

$= 60.$

$\det(x_3) = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 4 & 11 \\ 3 & -2 & 11 \end{bmatrix} = 88 - 24 - 33 - 48 + 44 + 33$

$= 60.$

$x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ ✓

135. ? $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = 1 \cdot \operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\det(A) = 1$

$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \end{bmatrix}$

try case: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{pmatrix} x_1 - x_2 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} 1 & a_1 & \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$ find $\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & b & c \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \end{pmatrix}^{-1}$

138 (b).

$$\begin{bmatrix} 1 & 2 & 4 \\ 5 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 29 \\ 10 \end{bmatrix}$$

$$\det(A) = 1 - 20 + 12 - 12 + 2 - 10$$
$$= -27$$

$$\det(x_1) = \begin{bmatrix} 31 & 2 & 4 \\ 29 & 1 & 2 \\ 10 & -1 & 1 \end{bmatrix} = 31 + 40 - 116 - 40 + 62 - 58$$
$$= -81$$

$$\det(x_2) = \begin{bmatrix} 1 & 31 & 4 \\ 5 & 29 & 2 \\ 3 & 10 & 1 \end{bmatrix} = 29 + 200 + 186 - 348 - 155 - 20$$
$$= -48 + 31 + 9$$
$$= -108$$

$$\det(x_3) = \begin{bmatrix} 1 & 2 & 31 \\ 5 & 1 & 29 \\ 3 & -1 & 10 \end{bmatrix} = 10 - 155 + 174 - 93 + 29 - 100$$
$$= -116 - 19$$
$$= -135$$

$$x = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

✓

139. (a) 1st cool formula.

$$-x_1 \cdot (x_1) + x_2 \cdot (-1 \cdot x_2) \cdots$$

$$= -x_1^2 - x_2^2 \cdots x_n^2$$

2nd.

$$\begin{array}{c|ccc} 0 & x_1 & \cdots & x_n \\ \hline x_1 & 1 & & 0 \\ \vdots & & 1 & \\ \vdots & 0 & & \\ x_r & & & 1 \end{array}$$

$$\det = 1 \cdot \det\left(-(x_1 \cdots x_n) \cdot 0 \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)$$

(b)

$$\det \begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{vmatrix} = a \cdot d^3 - bcd^2$$

$$\det \begin{vmatrix} a & 0 & 0 & b & 0 \\ 0 & a & b & 0 & 0 \\ 0 & c & d & 0 & 0 \\ c & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & d \end{vmatrix} = a \cdot (ad^3 - bcd^2) - b \cdot (c \cdot ad^2 - bcd)$$
$$= a^2d^3 - 2abcd^2 + b^2cd$$

$$\det \begin{vmatrix} \quad \end{vmatrix} = a(a^2d^3 - 2abcd^2 + b^2cd)$$

$$(ad - bc)^3$$

cool formula 1.

$$\det = d^3 \cdot \det\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} b & & 0 \\ & b & \\ 0 & & b \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{d} & & 0 \\ & \frac{1}{d} & \\ 0 & & \frac{1}{d} \end{pmatrix}\begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}\right)$$

$$d^3 \cdot \left(a - \frac{bc}{d}\right)^3$$

$$= (ad - bc)^3.$$

140. $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$

Lagrange formula.

Remove column when calculating determinant

143. $n=1$ $|\lambda+a_1| = \lambda+a_1$

$n=2$ $\begin{vmatrix} \lambda & -1 \\ a_2 & \lambda+a_1 \end{vmatrix} = \lambda^2 + \lambda a_1 - a_2$

$n=3$ $\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_3 & a_2 & \lambda+a_1 \end{vmatrix} = \lambda^2(\lambda+a_1) + a_3 + \lambda a_2.$

$= \lambda^3 + \lambda^2 a_1 + \lambda a_2 + a_3$

$\vdots$

induction

Induction:

$$A_n = \lambda A_{n-1} - (-1)\begin{vmatrix} 0 & -1 & & 0 \\ \vdots & \lambda & -1 & \\ & & \lambda & \ddots \\ & & & \ddots \\ a_n & & & \lambda+a_1 \end{vmatrix}$$

$$\Downarrow$$

$$\det\begin{vmatrix} 0 & -1 & & 0 \\ \vdots & \lambda & -1 & \\ & & \lambda & \ddots \\ a_n & & & a+a_1 \end{vmatrix} = (-1)^{n-2}\det\begin{vmatrix} -1 & 0 & 0 \\ \lambda & -1 & 0 \\ & \lambda & \ddots \\ & & a_n \end{vmatrix}$$

$= (-1)^{n-2}\cdot(-1)^{n-2}\cdot a_n$

$= a_n$

Transpose.

146.

$$\det(A) = n!\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & & n^2 \\ 1 & 2^4 & 3^4 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2^{2n-2} & 3^{2n-2} & & n^{2n-2} \end{vmatrix} = n!\begin{vmatrix} 1 & 1 & 1 \cdots & 1 \\ 1 & 4^2 & \cdots & 4^{n-1} \\ 1 & 9^2 & & 9^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & n^2 & \cdots & (n^2)^{n-1} \end{vmatrix}$$

133.

$$A^{-1} = \frac{1}{\det A} C^{T}$$

$$= \frac{1}{18} \cdot \qquad C = \begin{bmatrix} -5 & +1 & 7 \\ +7 & -5 & +1 \\ 1 & +7 & -5 \end{bmatrix}$$

$$C^{T} = \begin{bmatrix} -5 & +7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$$

$- \det ($

HW6.

150. invertible: $\forall\ k \in Z_p$, $\exists\ a$ s.t. $ak \equiv 1 \bmod p$.

$\because\ \gcd(k, p) = 1$

$\therefore$ By bazout therem, $\exists.\ sk + tp = 1$

$\exists.\ sk \equiv 1 \pmod p$

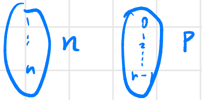
155. $$p^{\frac{(n+1)n}{2}}\ \underline{\times\ 0^{-}} \qquad p^{\underline{\frac{n(n-1)}{2}}}$$

\# function = # vectors in the space.

151

vector space: sets with elements/vectors/$\underline{\text{functions}}^{\text{(majority)}}$ that satisfy axioms.

$v \in Z_p^n$ $\overset{\text{function}}{\text{that}}$ maps $\{1 \sim n\}$ to Z_p (0, 1, 2 .. p-1)



152. $S = \{1, 2, \cdots, n\} \longrightarrow Z_p^n$ $\quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $x_1 \nearrow \in Z_p$ (0 ~ p-1)

linear map.

$\lambda A + \mu B$ of A, B : $V \to W$.

p^n vectors.

$\alpha A + \mu B, AB$: $Z_p^n \to Z_p^m$.

$$\mathrm{Hom}(Z_p^n, Z_p^m) = \{ax_1 + bx_2 \cdots kx_n \mid \} \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \overset{m \times n}{\boxed{}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$p^m \cdot p^n$

$\uparrow$ $p^m \cdot p^n$.

159.

$A: V \to W$

165. $\ker A: \{v \in V \mid Av = 0\}$

$\text{Graph} A: \{(v, w) \in V \oplus W : w = Av\}$

$V = \mathbb{R}^2$

$W = \mathbb{R}^2$.

$L: \mathbb{R}^n \longrightarrow \mathbb{R}$. $L: x \mapsto A \cdot x$

$\ker L =$

Sample MT:

P1: ① $ax^2+bxy+cy^2=1$

$(\sqrt{a}x+\frac{by}{2\sqrt{a}})^2+(c-\frac{b^2}{4a})y^2=1$

No.

$x^2+2xy+y^2=1$

$(x+y)^2=1$

$y=-x\pm1$

$B^2-4AC \begin{cases} <0 \to \text{ellipse.} \\ =0 \to \text{parabola.} \\ >0 \quad \text{hyperbola.} \end{cases}$

$A=C$ & $B=0$ circle.

② NO $T\begin{pmatrix}1 & -1\\ 2 & 2\end{pmatrix}$

③. Induction.

Sum $=-\frac{b}{a}$

product $=\frac{d}{a}$ even

$-\frac{d}{a}$ odd

$\underline{(-1)^5\cdot\frac{0}{1}}=0$ ✓

$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$

④. No. $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$

P2.

$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$

$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}^2=\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}$

$\det(\quad)=8-3=5$

P3.

$[y_1 \cdots y_n]\begin{bmatrix}b_{11}&\cdots&b_{1m}\\ \vdots & & \\ b_{n1}&\cdots&b_{nm}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$

$1\times n.$ $n\times m$ $m\times1$

$b.$

P4. $L\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}=4$

P5.

$\det(A)\cdot A^{-1}=\quad adj(A)$

$\det(A)\cdot\det(A^{-1})\cancel{=}\det(adj(A))$

$\det^2(A)$

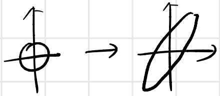
$\det(A)I=adj(A)\cdot A.$

$\det(\det(A)I)=\det(adj(A))\cdot\det(A)$

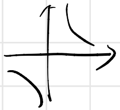
$(\det(A))^{n-1}=\det(adj(A))$

1.1. $x^2 + bxy + y^2 = 1$

$b \uparrow \quad (0, 2)$



$b > 2$



1.3 Vieta.

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$

$\sum_{j=1}^{n} r_1 r_2 \cdots r_{j-1} r_{j+1} \cdots r_n = \frac{a_1}{a_n}$

$- \frac{a_2}{a_n}$

$r_1 + \cdots + r_n = (-1)^n \frac{a_{n-1}}{a_n}$

3. bilinear form

$B(\vec{x}, \vec{y})$

$B(\lambda\vec{x}, \vec{y}) = \lambda B(\vec{x}, \vec{y})$

$B(\vec{x_1} + \vec{x_2}, \vec{y}) = B(\vec{x_1}, \vec{y}) + B(\vec{x_2}, \vec{y})$

Symetric: $S(x, y) = S(y, x)$

A ~ : $A(x, y) = -A(y, x)$

$B_{ij} = B(e_i, e_j) \qquad B^* = [B_{ij}]$

$B = \vec{x}^T \cdot B^* \cdot \vec{y}$

$B(e_1, e_1) = B\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right) = 2$

$B(e_1, e_2) = 2$

$B(e_2, e_1) = 0$

$B(e_2, e_2) = 0$

$B^* \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

$(x_1, x_2) \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1, x_2) \begin{pmatrix} 2y_1 + 2y_2 \\ 0 \end{pmatrix}$

$= 2x_1(y_1 + y_2)$

$S^* = S^{*T} \qquad A^* = -A^{*T}$

$S^* + A^* = B^*$

$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & - \\ - & 0 \end{bmatrix} + \begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$

e.g. $= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$S((x_1, x_2), (y_1, y_2)) = x^T \cdot S^* \cdot y$

$= (x_1, x_2) \begin{pmatrix} 2y_1 + y_2 \\ y_1 \end{pmatrix}$

$= x_1 (2y_1 + y_2)$

$+ x_2 y_1$

$A((x_1, x_2), (y_1, y_2)) = (x_1, x_2) \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}$

$= x_1 y_2 - x_2 y_1$

Soluton2:

$B = 2x_1(y_1 + y_2) = a_{11} x_1 y_1 + a_{12} x_1 y_2$

$a_{21} x_2 y_1 + a_{22} x_2 y_2$

Midterm Review

1.1. vector.

1.1.1 Cauchy-Schwarz: $\langle u,v\rangle^2 \le \langle u,u\rangle\cdot\langle v,v\rangle$

$$\langle v,w\rangle^2 \le \langle v,v\rangle\langle w,w\rangle$$

$$(|v|\cdot|w|\cdot\cos\theta)^2 \le |v|^2\cdot|w|^2$$

equal when $\theta = \pi + k\pi$.

$\Downarrow$

triangle inequality:

$$|u+v| \le |u|+|v|$$

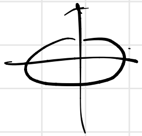
$$|u+v|^2 = \langle u+v, u+v\rangle$$

$$= \langle u,u\rangle+\langle v,v\rangle+\langle u,v\rangle+\langle v,u\rangle$$

$$= |u|^2+|v|^2+2\langle u,v\rangle$$

$$\le |u|^2+|v|^2+2\cdot|u|\cdot|v|$$

$$= (|u|+|v|)^2$$



1.2 Quadratic Curve

$$T_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$I = \cos\theta i + \sin\theta j$$

$$J = -\sin\theta i + \cos\theta J$$

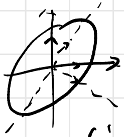
$$\underline{x}i+\underline{y}j = XI+YJ = (X\cos\theta - Y\sin\theta i)i + (X\sin\theta + Y\cos\theta)j$$

area: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad A(E) = \pi ab.$

$$x \to \left(\frac{b}{a}x\right)$$

$$\frac{\bar{x}^2}{b^2}+\frac{y^2}{b^2}=1$$

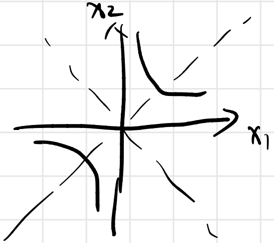
xy



$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2$$

$$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}e_1 - \frac{1}{\sqrt{2}}e_2$$

LT:



$x = {}^{x_1}(f_1) + x(f_2)$

$y = y_1 f_1 + y_2 f(2)$

$x^2 + xy + y^2 = 1$ $f_1\begin{pmatrix}1\\0\end{pmatrix} f_2\begin{pmatrix}0\\1\end{pmatrix}$

$$xy = \frac{\left(\frac{x+y}{\sqrt{2}}\right)^2}{2} - \frac{\left(\frac{x-y}{\sqrt{2}}\right)^2}{2}$$

Change of coordinates:

$x = Cx'$

$ax = a'x' = (aC)x' = (aC)x'$

$a' = aC$

LT: $\cancel{\underset{x \mapsto Ax}{y = Ax}}$ $A : R^n \rightarrow R^m$

$y = Dy'$ $Dy' = ACx'$

$x = Cx'$ $y' = D^{-1}ACx'$

$A' = D^{-1}AC$

Linear T: $y = cy'$ $y' = A'x'$

$x = cx'$ $cy' = ACx'$

$y' = c^{-1}ACx'$

vector space: set with elements/vectors/functions (congruity) that satisfy axioms.

↓ $V \in K^S$ $f(s) = K$

sub space: subset W in v.s. V that is closed under L.C. $au+av$

$R^3 \cong map(\{1,2,3\}, R)$

$\cong R \oplus R \oplus R$

(tuple).

↓ Category of vector spaces

linear functions/ forms: $V^* : V \to K$.

dual space.

$a(x) = a_1x_1 + \cdots a_nx_n$

evaluation: $[a_1 \cdots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}.$$

m-tuple of linear functions $a_1, \cdots a_m$ in n variables

linear map: $A: V \to W$ that respects Linearity

① $A(\lambda u + \mu v) = \lambda(Au) + \mu(Av)$

② injective.

Hom(V,W): set of all linear map forms a vector space

as satisfies.

O.D.E P12.

$$\begin{pmatrix} \frac{\sqrt{2}}{2}x_1 - \frac{3\sqrt{2}-\sqrt{2}x_1}{4} \\ \frac{\sqrt{2}x_1}{2} + \frac{3\sqrt{2}-\sqrt{2}x_1}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ \frac{3-x_1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{3\sqrt{2}x_1 - 3\sqrt{2}}{4} \\ \frac{\sqrt{2}x_1 + 3\sqrt{2}}{4} \end{pmatrix}$$

$3x_1 - 3$

$x_1 + 3$.

$x_2 = \frac{1}{3}x_1 + 4$.

$$\begin{matrix} \bar{x} \\ \bar{y} \end{matrix} \begin{pmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{x-y}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$2\bar{x}^2 + 2\bar{y}^2 = x^2 + y^2$$

$$\frac{\bar{x}^2}{2} - \frac{\bar{y}^2}{2} = xy$$

$$\frac{3}{2}\bar{x}^2 + \frac{5}{2}\bar{y}^2$$

$$B((u_1, u_2), (v_1, v_2)) = u_1v_1 - u_1v_2 + u_2v_1 - u_2v_2$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} x & 0 \\ 0 & -y \end{bmatrix} \quad \begin{bmatrix} ax & 0 \\ 0 & by \end{bmatrix}$$

LA. P44.

$$a(x) = \sum a_i x_i$$
$$= a_1 x_1 + \cdots + a_n x_n$$
$$a(\lambda x) = \sum a_i \lambda x_i$$
$$= a_1 \lambda x_1 + \cdots + a_n \lambda x_n$$
$$= \lambda a(x).$$

$$\longleftarrow$$

$$\left\{\begin{array}{ccc} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} & [a_1 \cdots a_n]. & \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ m\times 1 & 1\times n & n\times 1 \\ & & \\ \multicolumn{3}{l}{R^n \to R \to R^m} \\ \multicolumn{3}{l}{\longrightarrow} \end{array}\right.$$

$$AA^{-1} = 1$$
$$AB(B^{-1}A^{-1}) = I.$$

$$f(x,y) = xy$$

bilinear $f(x+a, y) = f(x,y) + f(a,y).$

$$f(x+y) = f(x) + f(y) ✗$$

$$f(x,y) = (x_1 + \cdots x_n)(y_1 + \cdots y_n)$$

$$f(x,x) = \sum_{ij} x_i x_j$$

HW 7.

177. $[A] = [v_1, v_2 \cdots v_n]$ lin. indep $\Leftrightarrow$ $x_1v_1 + \cdots x_nv_n = 0$ $\Leftrightarrow$ $x_1 \cdots x_n = 0$

if $Ax = 0$.

$A^{-1}A \cdot x = 0$.

$x = 0$.

178. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

dim(Image A) = dim (col space)

= Rank A = dim (row space).

$T: K^4 \to K^2$ = # lin. indep colums.

dim ~~= 2~~. dim (null(A)) = 2

dim (range A) = 4 − 2

(col A)

179.

2. if $n > 1$

180. $x^k = x_1^k L_1(x) + \cdots x_n^k L_n(x)$. $k = 0 \cdots m-1$

184. $\text{rk}(A+B) = \dim(\text{span}(a_1+b_1, \cdots a_n+b_n))$

$\text{rank}\, A = \dim(\text{span}(a_1, a_2 \cdots a_n))$

$\text{Rank}\, B = \dim(\text{span}(b_1 \cdots b_n))$

$\text{span}(a_1+b_1 \cdots a_n+b_n) \subseteq \text{span}(a_1 \cdots a_n) + \text{span}(b_1 \cdots b_n)$

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

185. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$

b^{-1} C.

$\begin{pmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$ $\left(\begin{array}{ccc|l} 2 & -1 & -1 & b_1 \\ 0 & \frac{3}{2} & -\frac{3}{2} & b_2 + \frac{1}{2}b_1 \\ 0 & 1 & 0 & b_3 + b_1 + b_2 \end{array}\right)$

187. $\begin{bmatrix} & A & \end{bmatrix}$: $m \times 2009$

unique solution: $\iff$ b in column space of A.

$\dim(\text{col } A) = n$.

~~(A invertible~~

Given $Ax = B$ & $(A|B)$

If rank A = rank $A|B$ = # rows in x : unique soln.

rank A = rank $A|B$ < # rows in x : ∞ many solutions

rank A < rank $A|B$: ✗ solution

↑ # rows without zeros

(a) ✓ $\ker A$ + rank A = $\dim V$

0 2009. 2009

(b) ✓

(c). Yes.

(d) No.

(e). Yes

(f) No

(g) No

(h). Yes

Sample: 2.

2. any $S \subseteq K^n$.

suppose basis $(e_1, e_2 \cdots e_p)$. can be completed to $(e_1, e_2 \cdots e_p, e_{p+1} \cdots e_n)$ to K^n.

$T: \mathbb{R}^n \to \mathbb{R}^n$. $T(e_i) = 0$, $1 \leq i \leq p$

$T(e_i) = 1$, $p+1 \leq i \leq n$

$S = \text{kernel}(T)$

$T: \mathbb{R}^m \to \mathbb{R}^n$ $m \geq n$ $V \subset \mathbb{R}^n$. basis $(e_1 \cdots e_p)$

195. (a). q^n

(b).

(c).

(d)

HW 8.

199. $\begin{bmatrix} 4 & 8 & 18 & 7 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 2.

$$\text{basis}\begin{cases} \text{col}: & \begin{pmatrix}4\\0\\0\\0\end{pmatrix}, \begin{pmatrix}8\\4\\0\\0\end{pmatrix} \\ \text{row}: & (4, 8, 18, 7) \quad (0, 4, 10, 1) \\ \text{null}: & \begin{pmatrix} -\frac{2t_1-5t_2}{4} \\ \frac{0-10t_1-t_2}{4} \\ t_1 \\ t_2 \end{pmatrix} = t_1\begin{pmatrix}-\frac{1}{2}\\ -\frac{10}{4} \\ 1 \\ 0\end{pmatrix} + t_2\begin{pmatrix}\frac{5}{4} \\ -\frac{1}{4} \\ 0 \\ 1\end{pmatrix} \end{cases}$$

197. $\left[\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & -2 & 4 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{cases} x_1 = -8 \\ x_2 = t+3 \\ x_3 = 2t+6 \\ x_4 = t \end{cases}$$

$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

201. $D_i(d) = \underbrace{\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & d & & \\ & & \uparrow & 1 & \\ 0 & & i & & 1 \end{bmatrix}}_{n\times n}$ $D^{-1}_i(d) = D_i(d^{-1})$

$L_{ij}(a) = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ -a & & & 1 \end{bmatrix}$ $i > j$, $i \rightarrow$ row, j column of $-a$

$L_{ij}^{-1}(a) = L_{ij}(-a)$

$\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$

?

200.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & -3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 0 & 5 & -3 & 7 & 3 \\ 0 & 15 & -9 & 21 & 2\lambda-1 \end{array}\right]$$

$$9 = 2\lambda - 1$$

$$\lambda = 5$$

Sample quiz 8

1. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

3. image: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ plane.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 1 & 2 & 1 & x_2 \\ 2 & 3 & 2 & x_3 \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} \qquad z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

202.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} & \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

0 0

①

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} .$$

$$(PP^T)_{ij} = \sum_{k=1}^{n} P_{ik} \cdot P^T_{kj} = \sum_{k=1}^{n} P_{ik} \cdot P_{jk} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise.} \end{cases}$$

② for I. S_{ij} row = S_{ji} column.

Suppose $P = S_1 S_2 \cdots S_k$. (elementary)

$$P^{-1} = (S_1 \cdots S_k)^{-1} = S_k^{-1} \cdot S_{k-1}^{-1} \cdots S_1$$

$$= S_k \cdots S_1$$

$$= S_k^t \cdots S_1^t$$

$$= (S_1 \cdots S_k)^t$$

$$= P^t$$

HW 9.

211. $x_1x_2 + x_2^2 = 0x_1^2 + 1x_1x_2 + 0x_2x_1 + 1x_2^2$

$$(x_1, x_2)\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

orthog: $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ ✗

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & -6 \\ 0 & -6 & 18 \end{bmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 6 \\ -6 \end{pmatrix} - \frac{14}{5} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Sample.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right] \quad L^{-1}$$

$M = LPU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}.$$

$-\frac{13}{11}$

$$u_2 = v_2 - \widehat{\text{proj}}(u_1)(v_2) = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} - \frac{98}{\overset{9}{}66} \cancel{\begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}} \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{11} \\ -\frac{10}{11} \\ -\frac{10}{11} \end{pmatrix} \quad \begin{pmatrix} \frac{9}{11} \\ \frac{3}{11} \\ -\frac{3}{11} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3 = v_3 - \widehat{\text{proj}}\, u_1(v_3) - \widehat{\text{proj}}\, u_2 v_3$$

$$= \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} - \frac{90}{66}\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} - \frac{6}{11}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

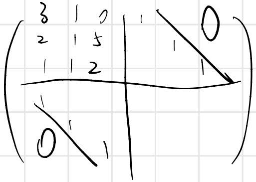
$\frac{15}{11}$

$$= \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} - \begin{pmatrix} \frac{15}{11} \\ \frac{60}{11} \\ \frac{77}{11} \end{pmatrix} - \begin{pmatrix} \frac{18}{11} \\ \frac{6}{11} \\ -\frac{6}{11} \end{pmatrix}$$

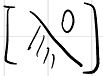
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$M = LPU \qquad P = L^{-1} M u^{-1}$

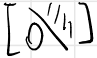
LPU:



$$\rightsquigarrow \left(\begin{array}{c|c} P & L^{-1} \\ \hline u^{-1} & \end{array}\right)$$

$L^{-1} =$ 

r_1 can modify r_2

$u^{-1} =$ 

c_1 modifies c_2 but not viceversa

$[\, M \mid I \,]$

$[\, L^{-1}M \mid L^{-1} \,]$

$[\, PU \mid I^{-1} \,]$

226.

anti- a $\Rightarrow$ $a_{ij} = -\overline{a_{ji}}$

$a_{ii} = -\overline{a_{ii}}$

Symmetrc $\Rightarrow$ $a_{ij} = a_{ji} = -\overline{a_{ji}}$

$a+bi = -(a-bi)$

$2a = 0$

[ ] all entries are Imaginary.

229.

$x = \sum x_i e_i \quad y = \sum y_j e_j$

$P(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{x}_i p_{ij} y_j, \quad p_{ij} = P(e_i, e_j)$

$D^* = \overline{D^t}$ $\quad v = Dv' \quad w = Cw'$

$P\langle v, w\rangle = \bar{v}^t P w$

$= \overline{v'^t}\, \overline{D^t}\, P C w'$

230,

232.

$A = \overline{A^t} \quad B = \overline{B^t}$

$(AB - BA) = \overline{B^t}\, \overline{A^t} - \overline{A^t}\, \overline{B^t}$

$= \overline{(AB)^t} - \overline{(BA)^t}$

$= \overline{(AB - BA)^t}$

Hermitian → anti - Hermitian

Sample Quiz 9

1. ~~False~~. True Hermitian:

sesquilinear $H(av, bw) = \bar{a} b H(v, w)$

$H(v, w) = \overline{H(w, v)}$

$H(v, v) = \overline{H(v, v)} \in R$

$H_{ij} = \overline{H_{ji}}$

2. False. anti-symmetric: $A_{ij} = -A_{ji}$

Imaginary $\overline{A_{ij}} = -\cancel{\overline{A_{ij}}}$

$-A_{ij} = \overline{A_{ij}}$

$A_{ij} = \overline{A_{ji}}$

3. True $H(v, v) \in R$

4. False $A = \overline{A^t}$ $B = \overline{B^t}$

$AB = \overline{A^t B^t}$

$= \overline{(BA)^t}$

1. $A = \overline{A^t}$

$iA = i \overline{A^t}$

$= \overline{-iA^t}$

$= -\overline{(iA)^t}$

2. similar.

3. in HW.

HW.

233. $H(z_1, z_2) = \bar{z}_1 z_2 - \bar{z}_2 z_1 = (\bar{z}_1, \bar{z}_2) \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{m} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ anti-hermitian

$\bar{z}_1 z_2 + \bar{z}_2 z_1$

$M = -m^*$

$M = i \cdot H = i \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{H}$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

Need to find

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$

$Z = A\tilde{Z} \quad Z^* = \tilde{Z}^* A^*$

s.t. $Q = Z^* M \cdot Z$

$= \tilde{Z}^* A^* M \cdot A \cdot \tilde{Z}$

$= \tilde{Z}^* \tilde{M} \tilde{Z}$

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

③ $\begin{bmatrix} 1 & i \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$|z_1|^2 - 2|z_2|^2.$

$\tilde{M} = A^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A \qquad A = \begin{pmatrix} \overset{v_1}{|} & \overset{v_2}{|} \end{pmatrix}$

$A^* = \begin{pmatrix} \overline{\quad v_1^* \quad} \\ \overline{\quad v_2^* \quad} \end{pmatrix}$

$\tilde{M}_{ij} = v_i^* \cdot M \cdot v_j.$

Try ...

$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \rightarrow (1, -i) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 2i \Rightarrow \lambda_1$

$v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \rightarrow ($

$-2i \Rightarrow \lambda_2$

233. ①.

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Herm

$$m \sim i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

235.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\Delta_0 = 1$

$\Delta_1 = 1$

$\Delta_2 = 1$

$\Delta_3 = -1 \cdot$

$\Delta_4 = -1 \cdot 1 \cdot -1 = 1$

$p = 2$

$q = 2$

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Swap 1st 4th row/column

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{2} & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$\Delta_0 = 1$

$\Delta_1 = 1$

$\Delta_2 = -2$

$\Delta_3 = -2$

$\Delta_4 = \frac{1}{2} \cdot -1 \times \frac{1}{2}$

$(3, 1)$

$\bar{z}_1 z_2 + \bar{z}_2 z_1$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ H.

$$\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$$

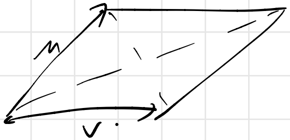
$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

253. $\langle u, v\rangle = \langle v, u\rangle = 0$

$$\Rightarrow \|u\|^2 + \|v\|^2 = \langle u, u\rangle + \langle v, v\rangle$$
$$= \langle u-v, u-v\rangle$$
$$= \|u-v\|^2 \ .$$

$$\langle u, u\rangle + \langle v, v\rangle = \langle u-v, u-v\rangle$$
$$\langle u, v\rangle = -\langle v, u\rangle$$

254.



255.

Sample Midterm 2.

1. Given hermitian form, diagonalize.

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

More efficient?

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

补 $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$\begin{pmatrix} 0 & i \\ -i & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -i & 2 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{i}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & -\frac{i}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2、 $\lambda = 1, 2, -1$

$\lambda = 1$

$$T - \lambda I = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$x = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}, k \in F.$$

$a = 2$

$$T - \lambda I = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$x = -1$

$$I - \lambda I = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}.$$

$$x = \begin{pmatrix} \frac{3}{2} \\ 1 \\ -1 \end{pmatrix}$$

3、

$$-\det A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$\det A = -1$

$\det(A^{-1}) = 1$?

4.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \underset{\sim\sim\sim}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}$$