Final (Canez)

1. (a) two ways to find inverse.

① $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$ ↘ with ± 1

$$\operatorname{adj} A = \begin{pmatrix} -24 & 38 & 12 \\ -16 & -20 & -6 \\ 4 & 6 & 2 \end{pmatrix}^t$$

$$\det A = -16 - 2\times(-6) = -4$$

$$A^{-1} = \begin{pmatrix} -6 & 4 & -1 \\ -\frac{19}{2} & 5 & -\frac{3}{2} \\ -3 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

② Elementary operation

$$(A \mid I) \rightarrow (E, P)$$

$P_1 P_2 \cdots P_m A = E \qquad P_1 P_2 \cdots P_m = P$

$E = I \longrightarrow P = A^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 6 & -20 & 3 & 0 & 1 \end{array}\right)$$

remember: $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is $\underline{\text{row}_2} - \text{row}_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -\frac{19}{2} & 5 & -\frac{3}{2} \\ 0 & 0 & 1 & -3 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

(b) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 & 4 & -1 \\ -\frac{19}{2} & 5 & -\frac{3}{2} \\ -3 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$

2. $u, v \in W$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$u+v \notin W$

consider 0 matrix

3. $a_1(x+1) + a_2(2x^2+3) + a_3(3x-5) = 0$

$2a_2 x^2 + (a_1+3a_3)x + (a_1+3a_2-5a_3) = 0$ $\Leftrightarrow$ coordinate $\begin{pmatrix} 1 & 3 & -5 \\ 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$

$a_2 = 0$ $a_1 = -3a_3$. $a_3 = 0$.

Yes.

Lin. indp $\Leftrightarrow$ echelon form has pivot in each column.

Yes # dim V independent vectors span V.

4. 1. ref

2. row space : rows after ref

col space : original columns for pivot

null space : plug in t.

$$\begin{pmatrix} 1 & 1 & -3 & 3 & 5 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Null: an arbitrary vector in null space has form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s+4t \\ s \\ t \\ -2t \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

5. (a)

$$T(A+B) = AX+BX-XA-XB$$
$$= AX-XA+BX-XB$$
$$= TA+TB$$

$$T(\lambda A) = \lambda AX - \lambda XA$$
$$= \lambda(T(A))$$

Find Null T

$A \in$ Null T

$[T][A]=0$

$[A'] = 0$

$\Rightarrow$ find relation between elements of A'

(b). suppose $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in$ Null T

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a_{11} & 2a_{11}-a_{12} \\ a_{21} & 2a_{21}-a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+2a_{21} & a_{12}+2a_{22} \\ -a_{21} & -a_{22} \end{pmatrix}$$

$$\begin{pmatrix} -2a_{21} & 2a_{11}-2a_{12}-2a_{22} \\ 2a_{21} & 2a_{21} \end{pmatrix} = 0$$

$\Rightarrow a_{21}=0$, $a_{11} = a_{12}+a_{22}$

$\Rightarrow$ A has form

$$\begin{pmatrix} a_{12}+a_{22} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

$$= a_{12}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6.

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

$u = \langle u_1, u_2\rangle$ $\quad \|u\| = |\langle u_1, u_2\rangle| \leq \|u_1\|\|u_2\|$

$v = \langle v_1, v_2\rangle$

$$|\langle u_1+v_1, u_2+v_2\rangle| \leq \cancel{\|u_1+v_1\|}\|u_2+v_2\|$$

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle$$

$$\langle v, u\rangle = \overline{\langle u, v\rangle}$$

$$= \langle u, u\rangle + \langle v, v\rangle + 2\,\mathrm{Re}\langle u, v\rangle$$

$$\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle|$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$$

$$= (\|u\| + \|v\|)^2$$

7. $x = c_1 v_1 + \cdots c_n v_n$.

want to $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \langle x, v_1\rangle \\ \vdots \\ \langle x, v_n\rangle \end{pmatrix}$

$\Leftrightarrow x = \langle x, v_1\rangle v_1 + \cdots \langle x, v_n\rangle v_n$

$\langle x, v_i\rangle = c_i$

8. ① formula.

$-6 + 6 - 24 + 3 = -21$

② cofactor.

$2 \times (-9) + 1 \times (-3) = -21$

(b) non zero $\Leftrightarrow$ invertible.

9.

char polynomial.

$$\det(\lambda I - T) = \begin{vmatrix} \lambda-2 & -2 & 1 \\ -1 & \lambda-3 & -1 \\ -1 & -2 & \lambda-2 \end{vmatrix}$$

Quick way to find eigenvalue?

$$= (\lambda-2)(\lambda^2-5\lambda+6) - (-2\lambda+2) - (\lambda-1)$$

$$= (\lambda-1)[(\lambda-2)(\lambda-4)-2-1]$$

$$= (\lambda-1)(\lambda^2-6\lambda+5)$$

$$= (\lambda-1)^2(\lambda-5)$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1+2x_2+x_3 \\ x_1+3x_2+x_3 \\ x_1+2x_2+2x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}$$

W_1: $\lambda = 1$

$$x_1+2x_2+x_3=0$$
$$x_1+2x_2+x_3=0$$
$$x_1+2x_2+x_3=0$$

$$\begin{pmatrix} -2s-t \\ s \\ t \end{pmatrix}$$

W_5: $\lambda = 5$

$$-3x_1+3x_2+x_3=0$$
$$x_1-2x_2+x_3=0$$
$$x_1+\cancel{2x_3}-3x_3=0$$

$$4x_1 = 5x_2.$$

10.

$$\det(\lambda I - T)$$

$$= \begin{vmatrix} \lambda-4 & \lambda & \lambda+2 \\ \lambda-2 & \lambda-5 & \lambda-4 \\ \lambda & \lambda & \lambda-5 \end{vmatrix}$$

$=$

11. (a) For ξe^{rt} a solution

$$\Rightarrow \quad (\xi e^{rt})' = A \cdot \xi e^{rt}$$

$$\xi \cdot r \cdot e^{rt} = A \cdot \xi e^{rt}$$

$$r \cdot \xi = A \xi$$

(b) Step 1: find eigenvalues:

$$\det(\lambda - A) = \begin{vmatrix} \lambda-4 & -2 \\ -1 & \lambda-3 \end{vmatrix} = 2 - \lambda^2 - 12 + 7\lambda$$

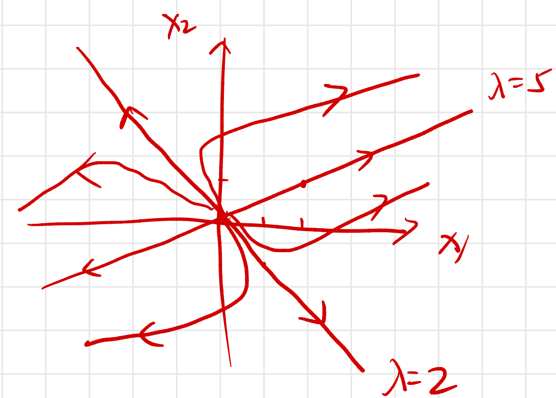
$$= -(\lambda-2)(\lambda-5)$$

$$\lambda = 2, 5$$

Step 2: find $\vec{x}$

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 + 2x_2 \\ x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \quad \vec{x_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix} \quad \vec{x_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



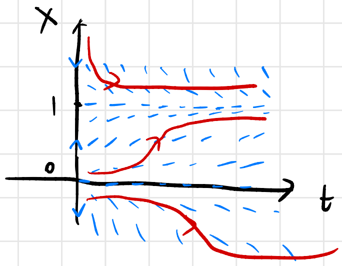
Step 3: combine.

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Phase Portrait

1. Slope Field

$$\frac{dx}{dt} = x(1-x)$$

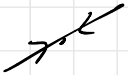


2. Phase plane (for system)

Critical points: were $x_1' = 0$ $x_2' = 0$.

$\lambda > 0$ grow

< 0 decay



① 2 real λ

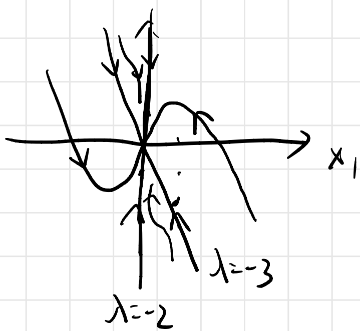
- both +

  see 11. (b) case 2. unstable

- both −

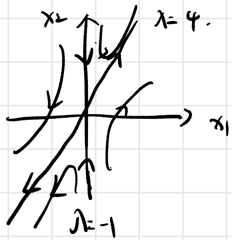
  e.g. $\lambda_1 = -3$ $\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

  $\lambda_2 = -2$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



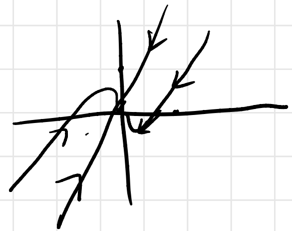
Stable.

\- one + one -



saddle

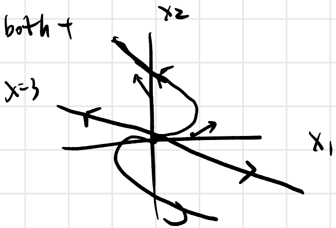
\- both -



stable

② Repeated eigenvalues.

\- both +



pick 2 points, say $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\ \frac{1}{3}\end{pmatrix}$$

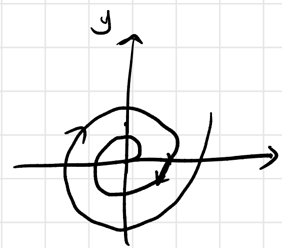
$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-3\\4\end{pmatrix}$$

unstable

③ complex.

\- Re -

$\lambda = -2 \pm 3i$



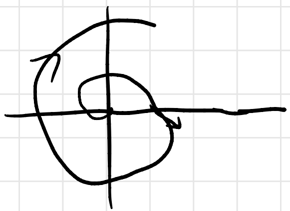
Stable spiral

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2\\-3\end{pmatrix}$$

- Re +

$\lambda = 2 \pm 3i$

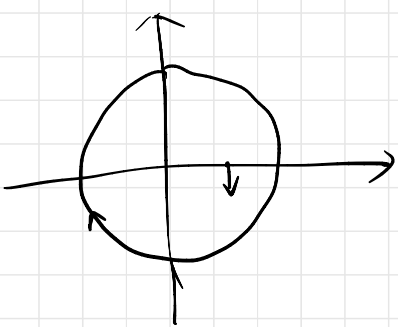
unstable spiral



$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$

- no Re.

$\lambda = \pm\sqrt{5}\,i$ center



$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-5\end{pmatrix}$

1-(a) Fundamental matrix

1. Find eigenvalues.

$$\begin{vmatrix} \lambda-6 & -5 \\ -2 & \lambda+3 \end{vmatrix} = \lambda^2-3\lambda-18-10=(\lambda-7)(\lambda+4)$$

$\lambda_1=7 \Rightarrow$ $\begin{array}{l} 6y+5z=7y \\ 2y-3z=7z \end{array}$ $\vec{x}_1=\begin{pmatrix}5\\1\end{pmatrix}$

$\lambda_2=-4 \Rightarrow$ $\vec{x}_2=\begin{pmatrix}1\\-2\end{pmatrix}$

$$\vec{x}=c_1e^{7t}\begin{pmatrix}5\\1\end{pmatrix}+c_2e^{-4t}\begin{pmatrix}1\\-2\end{pmatrix}$$

FM: $\psi(t)=\begin{pmatrix}5e^{7t} & e^{-4t} \\ e^{7t} & -2e^{-4t}\end{pmatrix}$

SFM satisfy $\Phi(0)=I$

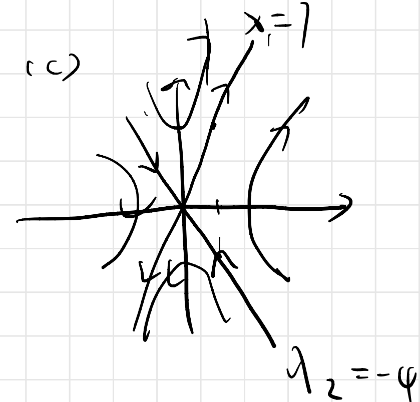
$\Phi(t)=\psi(t)\cdot\psi(0)^{-1}$ $\begin{pmatrix}5 & 1\\ 1 & -2\end{pmatrix}^{-1}$

$=\begin{pmatrix} & \\ & \end{pmatrix}\cdot\frac{1}{11}\cdot\begin{pmatrix}-2 & -1\\ -1 & 5\end{pmatrix}$ $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\cdot\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$.

$= \cdots$

(b) $\vec{x}(t)=\Phi(t)\cdot \cancel{c}\ x(0)$

(c)



Fundamental matrix.

Def: ① invertible.

② $\dot{\psi}(t) = A \cdot \psi(t)$.

$\Longleftrightarrow$. n lin. indep columns, each a solution.

gen. soln $\vec{x} = \psi(t) \cdot c$

$\psi(t) \cdot$ non singular constant matrix. is still fundamental.

13. (a) $\dot{\vec{x}} = A \cdot \vec{x}$

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} \cdots$$

$$= I + \sum_{m=1}^{\infty} \begin{pmatrix} \frac{d_1^m \cdot t^m}{m!} & & 0 \\ & \ddots & \\ 0 & & \frac{d_n^m t^m}{m!} \end{pmatrix}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{d_1 t} = 1 + d_1 t + \cancel{+} \frac{d_1^2 t^2}{2!} + \cdots$$

$$= \begin{pmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{pmatrix}$$

(b) $A = S \Lambda S^{-1}$

$$A^k = S \Lambda^k S^{-1}$$

$$e^{At} = I + \sum_{n=1}^{\infty} \frac{A^n \cdot t^n}{n!}$$

$$= I + \sum_{n=1}^{\infty} \frac{S\Lambda^n S^{-1} \cdot t^n}{n!}$$

$$= SIS^{-1} + \sum_{n=1}^{\infty} \frac{S(\Lambda \cdot t)^n S^{-1}}{n!}$$

$$= S \left(I + \sum_{n=1}^{\infty} \frac{(\Lambda t)^n}{n!} \right) S^{-1} = S e^{\Lambda t} S^{-1}.$$

(c) $e^{\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} t} = S e^{\Lambda t} S^{-1}$

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\frac{1}{3} \cdot \qquad \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$e^{At} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

14.

(a) $\dfrac{d(u+iv)}{dt} = A\cdot(u+iv)$

$\vec{x} = c_1 \begin{pmatrix} \\ \end{pmatrix} + c_2 \begin{pmatrix} \\ \end{pmatrix}$

$u' + iv' = Au + iAv$

$u' = Au$

$v' = Av.$

(b) $\begin{vmatrix} \lambda-5 & -10 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0.$

$(\lambda-1)^2 = -4$

$\lambda = 1 \pm 2i$

$\lambda_1 = 1+2i$

$\begin{pmatrix} 5 & 10 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 5y+10z \\ -2y-3z \end{pmatrix} = \begin{pmatrix} y + i2y \\ z + i2z \end{pmatrix}$

$10z = (-4+2i)y$

$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -4+2i \end{pmatrix}$

$z(t) = e^{1+2it} \begin{pmatrix} 10 \\ -4+2i \end{pmatrix}$

$= e^{t} \cdot (\cos 2t + i \sin 2t) \begin{pmatrix} 10 \\ -4+2i \end{pmatrix}$

$= e^{t} \begin{pmatrix} 10\cos 2t \\ -4\cos 2t - 2\sin 2t \end{pmatrix} + ie^{t} \begin{pmatrix} 10 \sin 2t \\ -4\sin 2t + 2\cos 2t \end{pmatrix}$

Fundamental sets of complex eigenvector soln

Set up: $x' = Ax$ (1)

$Av = \lambda v$ w/ $\lambda = \alpha + i\beta \in \mathbb{C}$

$v = u + iw \in \mathbb{C}^n$

① linearly indep. complex solutions to (1): $(F = \mathbb{C})$

$$z(t) = e^{\lambda t} v \qquad \bar{z}(t) = e^{\bar{\lambda} t} \bar{v}$$

take
$$\begin{aligned} z(t) &= e^{(\alpha + i\beta)t}(u + iw) \\ &= e^{\alpha t} \cdot (\cos\beta t + i \sin\beta t)(u + iw) \\ &= e^{\alpha t}(u\cos\beta t - w\sin\beta t) + \\ &\quad i e^{\alpha t}(u\sin\beta t + w\cos\beta t) \\ &= \operatorname{Re} z(t) + i \operatorname{Im} z(t) \end{aligned}$$

② lin. indep Real solution of (1): $F = \mathbb{R}$

$$x_1(t) = \operatorname{Re} z(t)$$
$$x_2(t) = \operatorname{Im} z(t)$$

15. (a) $\xi e^{rt} + r\xi t e^{rt} = A(\xi t e^{rt} + \eta e^{rt})$

$+ r\eta e^{rt}$.

$A\xi = r\xi$.

$\xi + r\xi t + r\eta = A(\xi t + \eta)$

$\xi = \cancel{r\xi t} + (A - rI)\eta$

(b)

$$\begin{vmatrix} \lambda+3 & 2 \\ -2 & \lambda+7 \end{vmatrix} = \lambda^2 + 10\lambda + 25 = (\lambda+5)^2 = 0$$

$\lambda_1 = \lambda_2 = -5$

$$\lambda I - A = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\vec{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t}$

$2y - 2z = 1$

$(A - -5I)\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \eta = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

$x = \cancel{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-5t} + c_2 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}} e^{-5t}$

$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-5t} + \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} e^{-5t} \right)$

16 !??

17. (a).

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

$$\langle f, g\rangle = \int_{-L}^{L} f(x) g(x)\,dx$$

$$\left\langle \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\right\rangle = \begin{cases} 0, & m \neq n \\ L, & m = n \\ 2L, & m = n = 0 \end{cases}$$

$$\left\langle \cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right\rangle = 0 \quad \text{for } \forall\, m, n$$

$$\left\langle \sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right\rangle = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$\Rightarrow$ when $m = n = 0$?

$$a_0 : \langle f(x), \cos(0)\rangle = \left\langle \left(\frac{a_0}{2} + \sum_{m=1}^{m}\left(a_m \cos\frac{\pi x m}{L} + b_m \sin\frac{\pi x m}{L}\right)\right), \cos 0\right\rangle$$

$$= \frac{a_0}{2}\langle \cos 0, \cos 0\rangle + 0$$

$$= \frac{a_0}{2} \cdot 2L$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) \cdot 1\, dx$$

$n > 0$

$$a_n : \left\langle f(x), \cos\frac{n\pi x}{L}\right\rangle$$

$$= \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cdot \cos\frac{m\pi x}{L} + b_m \cdot \sin\frac{m\pi x}{L}\right), \cos\frac{n\pi x}{L}\right\rangle$$

$$= \frac{a_0}{2}\left\langle \cos(0), \cos\frac{n\pi x}{L}\right\rangle + \left\langle a_n \cdot \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\right\rangle$$

$$\Downarrow$$

$$= 0 + a_n \cdot L$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \cos\frac{n\pi x}{L}\, dx.$$

b_n :

$$\left\langle f(x),\ \sin\frac{n\pi x}{L} \right\rangle$$

$$= \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right),\ \sin\frac{n\pi x}{L} \right\rangle$$

$$= b_n \cdot L$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

(b)

$$\langle f(x), \cos 0 \rangle$$

$$= a_0 \cdot 2L$$

$$a_0 = \frac{1}{2L} \int_{-2}^{2} f(x) \cdot dx$$

$$= \frac{1}{2L} \int_0^2 (1-x)\, dx$$

$$= \frac{1}{2L} \; x - \frac{1}{2}x^2 \Big|_0^2$$

$$= 0$$

$$\langle f(x), \cos \frac{n\pi x}{L} \rangle = a_n \cdot L$$

$$a_n = \frac{1}{L} \int_{-2}^{2} f(x) \cdot \cos \frac{n\pi x}{L} \cdot dx$$

$$= \frac{1}{L} \cdot \int_0^2 (1-x) \cdot \cos \frac{n\pi x}{2}\, dx$$

$$= \frac{1}{2} \left((1-x) \cdot \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \Big|_0^2 + \int_0^2 \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right)$$

$$= -\frac{1}{n\pi} \; \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \Big|_0^2$$

$$= \frac{2}{(n\pi)^2} \left(1 - \cos n\pi \right)$$

$$\langle f(x), \sin \frac{n\pi x}{L} \rangle = b_n \cdot L$$

$$b_n = \frac{1}{L} \int_0^2 (1-x) \cdot \frac{\sin n\pi x}{L} \cdot dx$$

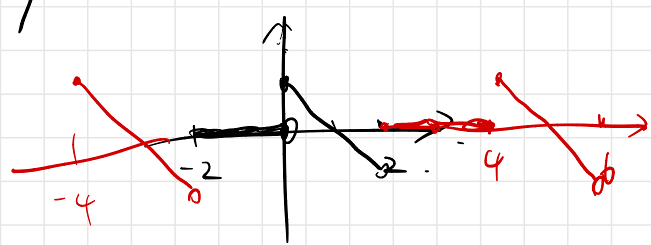
$$= -\frac{1}{2} \cdot (1-x) \cdot \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \Big|_0^2 + \frac{1}{2} \int_0^2 \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \cdot dx$$

$$= -\frac{1}{n\pi} (1-x) \cos \frac{n\pi x}{2} \Big|_0^2 + \frac{1}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} \cdot dx$$

$$= -\frac{1}{n\pi} \left(-1 \cdot \cos n\pi - 1 \right) + \frac{2}{(n\pi)^2} \sin n\pi$$

$$= \frac{1}{n\pi} \left(\cos n\pi + 1 \right)$$

$$\sum_{n=1}^{\infty} \left(\frac{2}{(n\pi)^2} (1 - \cos n\pi) \cdot \cos \frac{n\pi x}{2} + \frac{1}{n\pi} (\cos n\pi + 1) \sin \frac{n\pi x}{2} \right)$$



18. (a) odd extension of $f(x)$.

$$f_{odd}(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0 \end{cases}$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

$$= 2 \int_0^1 (1+x) \sin n\pi x \, dx$$

$$= -\frac{2}{n\pi} \cdot (1+x) \cos n\pi x \Big|_0^1 + 2 \int_0^1 \frac{\cos n\pi x}{n\pi} dx$$

$$= -\frac{2}{n\pi} \cdot (2\cos n\pi - 1) + \frac{2}{n\pi} \cdot \frac{\sin n\pi x}{n\pi} \Big|_0^1 .$$

$$= \frac{2}{n\pi} (1 - 2\cos n\pi)$$

$$f = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - 2\cos n\pi) \cdot \sin n\pi x$$

(b) even extension

$$f_{even}(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0. \end{cases}$$

$$= \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx.$$

$$= 2 \cdot \int_0^1 (1+x) \cdot \cos n\pi x \cdot dx$$

$$= 2 \cdot (1+x) \frac{\sin n\pi x}{n\pi} \Big|_0^1 - 2 \int_0^1 \frac{\sin n\pi x}{n\pi} \cdot dx$$

$$= +2 \cdot \frac{1}{n\pi} \cdot \frac{\cos n\pi x}{n\pi} \Big|_0^1 .$$

$$= \frac{2}{(n\pi)^2} \cdot (\cos n\pi - 1)$$

$$\langle f(x), a_0 \rangle = a_0 2L.$$

$$a_0 = \frac{1}{2L} \cdot 2 \cdot \int_0^L f(x) \cdot dx$$

$$= \frac{1}{L} \int_0^L f(x) \cdot dx.$$

$$= \int_0^1 (1+x) \cdot dx = \frac{x^2}{2} + x \Big|_0^1 = \frac{3}{2}$$

$$f = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} (\cos n\pi - 1) \cdot \cos n\pi x .$$

19.

$$u(t) = g(x) \cdot f(t).$$

$$4 \cdot (\partial x \partial x\, g(x))\, f(t) = g(x) \cdot (\partial t \cdot f(t))$$

$$\frac{4 \cdot (\partial xx \cdot g(x))}{g(x)} = \frac{(\partial t \cdot f(t))}{f(t)} = \lambda$$

$$\begin{cases} \partial xx \cdot g(x) = \frac{\lambda}{4}\, g(x) \\ \partial t\ f(t) = \lambda\ f(t) \end{cases}$$

Fall 2014 Finals.

P4 (1) $y'' - 2y' - 3y = 0$.

(quick way) plug in $y_{(x)} = e^{\lambda x}$

$$(\lambda^2 - 2\lambda - 3)e^{\lambda x} = 0.$$

$$\lambda = 3/-1$$

$$y_{(x)} = c_1 e^{3t} + c_2 e^{-1t}.$$

2nd order ODE $\Rightarrow$ 2 initial conditions.

(2) $y'' - 2y' - 3y = 10\cos(t)$

Strategy: find a particular sol'n then add general sol'n
(Recall ex 3.7.3 in ODE)

$$10\cos(t) = 10 \cdot \left(\frac{e^{it} + e^{-it}}{2} \right)$$
$$= 5e^{it} + 5e^{-it}.$$

Find $y_1(t)$ s.t.

$$P = \left[\left(\frac{d}{dt}\right)^2 - 2\frac{d}{dt} - 3 \right]$$

$$P\, y_1(t) = 5 \cdot e^{it}$$

$$P\, y_2(t) = 5e^{-it}$$

then $y_1(t) + y_2(t)$ will be a particular sol'n

$y_1(t) = c_1 e^{it}$, plug in

(*) $(i^2 - 2i - 3)\, c \cdot e^{it} = 5 \cdot e^{it}$

$$c_1 = \frac{5}{-1-2i-3} = \frac{5}{-4-2i}$$

$$y_2(t) = c_2 \cdot e^{-it}$$

$$(-1+2i-3)\, c_2 \cdot e^{-it} = 5 \cdot e^{-it}$$

$$c_2 = \frac{5}{-4+2i}$$

$y_1(t) + y_2(t) = \frac{5}{-4-2i} e^{it} + \frac{5e^{-it}}{-4+2i}$

Gen: $y_1(t) + b_1 e^{-t} + b_2 e^{3t}$, b_1 b_2 free

5) basis of sol'n of equation

$$y'(t) = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} y(t)$$

• Jordan decomposition of A

$\det(A-\lambda) = 0 \qquad \lambda_1 = 3$
$\qquad\qquad\qquad\qquad \lambda_2 = -1$

$\ker(A-\lambda_1) = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$
$\ker(A-\lambda_2) = \operatorname{span}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_2$

Let $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ then $A \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$$A = C \cdot \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}_{J} C^{-1}$$

e^{At} · (column space of e^{At} is gen sol'n)

basis:

$$e^{At} = C \cdot e^{Jt} \cdot C^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} C^{-1} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{3t}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}\right] C^{-1}$$

2)

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\lambda_1 = -1$

$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\lambda_2 = 2$.

$v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\lambda_3 = 0$.

$C = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$A \cdot C = C \cdot \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

$A = C \cdot \begin{pmatrix} 1 & & \\ & 2 & \\ & & 0 \end{pmatrix} C^{-1}$

C^{-1} block diagonal

$C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

6).

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u.$$

set $u(t, x) = g(t) \cdot f(x)$

$\left(\frac{\partial}{\partial t}\right)^2 u = (\partial_t^2 g) \cdot f(x)$

$(\partial_x^2) u = g \cdot (\partial_x^2 f)$

$(\partial_t^2 g) \cdot f = g \cdot (\partial_x^2 f) \cdot g \cdot f$.

$\frac{(\partial_t^2 g)}{g} = \frac{(\partial_x^2 f)}{f} + 1 \overset{\text{set}}{=} \lambda$.

$\begin{cases} \partial_t^2 g = \lambda \cdot g \\ \partial_x^2 f = (\lambda - 1) f \end{cases}$

e^{rt}

$(r^2 - \lambda) e^{rt} = 0$

$r = \pm\sqrt{\lambda}$

$\Rightarrow$ $g = e^{\sqrt{\lambda} t} c_1 + e^{-\sqrt{\lambda} t} c_2$.

$f = e^{\sqrt{\lambda - 1} x} c_3 + e^{-\sqrt{\lambda - 1} x} c_4$

gen sol'n $\sum_\lambda c_\lambda \cdot g_\lambda(t) \cdot f_\lambda(x)$.

P7) 1).

$$|x| = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos(mx) + b_m \sin(mx))$$

$$\langle f(x), \cos nx \rangle = a_n \cdot \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cdot d\left(\frac{\sin(nx)}{n}\right) \cdot dx$$

$$= \frac{2}{\pi} \left(x \cdot \frac{\sin(nx)}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} \cdot dx \right)$$

0

$$= -\frac{2}{\pi} \cdot \frac{1}{n} \cdot \frac{\cos nx}{-n} \Big|_0^{\pi}$$

$$= \frac{2}{n^2 \pi} \left((-1)^n - 1 \right)$$

integration by parts

$$\int u \cdot dv = uv - \int v \, du$$

$$\int_a^b u(x) v'(x) dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x) v(x) dx$$

$$\int_{-\pi}^{\pi} |x| \cos(n_0 x) dx$$

$$= 2 \cdot \int_0^{\pi} x \cdot \cos(n_0 x) dx$$

$$= 2 \cdot \int_0^{\pi} x \cdot d\left(\frac{\sin(n_0 x)}{n_0}\right) dx$$

$$= 2 \cdot x \cdot \frac{\sin n_0 x}{n_0} \Big|_{x=0}^{x=\pi} - 2 \int_0^{\pi} \frac{\sin(n_0 x)}{n_0} dx$$

$$= -2 \int_0^{\pi} \frac{\sin(n_0 x)}{n_0} dx$$

$$= -\frac{2}{n_0} \left(\frac{\cos n_0 x}{-n_0} \right) \Big|_0^{\pi}$$

$$= + \frac{2}{n_0^2} \left((-1)^{n_0} - 1 \right)$$

$$\cos(n_0 \pi) = (-1)^{n_0}$$

$$\pi a_n = \frac{2}{n_0^2} \left((-1)^{n_0} - 1 \right)$$

$$a_n = \frac{2}{\pi \cdot n_0^2} \left((-1)^{n_0} - 1 \right)$$