Lecture 3

~~Last time~~

$X_1, \dots, X_n \sim P_\eta$, $\eta \in H \leftarrow$ subset of a normed ~~line~~ space.

$P_\eta \ll \mu$ having density p_η.

~~Second~~ First order Frechet Taylor expansion:

$$\chi(\eta) = \chi(\hat{\eta}) + \chi'_{\hat{\eta}}(\eta - \hat{\eta}) + o_P(\|\eta - \hat{\eta}\|^2)$$

$$\chi'_{\hat{\eta}}(\eta - \hat{\eta}) = \int \chi^{(1)}_{\hat{\eta}} (dP_\eta - dP_{\hat{\eta}})$$

$$= \int \chi^{(1)}_{\hat{\eta}} \, dP_\eta \longrightarrow \text{estimate with } \mathbb{P}_n \chi^{(1)}_{\hat{\eta}}$$

Alternative ~~the~~ way of looking at ~~$\mathbb{P}_n$~~ $\chi^{(1)}_{\hat{\eta}}$ is the following.

Consider the model $\mathcal{P} = \{P_\eta : \eta \in H\}$. A first order tangent set at P_η is the set of all $\dot{g}_\eta$ s.t.

$$\dot{g}_\eta = \frac{\partial}{\partial t} p_{\eta_t}\Big|_{t=0} \Big/ p_\eta \quad \text{(scores at } P_\eta \text{ corresponding to the path)}$$

for some $\{P_{\eta_t} : t \in (-1, 1)\} \subset \mathcal{P}$ & $P_{\eta_0} = P_\eta$.

$(t \to \eta_t \to P_{\eta_t})$

then, $\chi^{(1)}_\eta$ ~~satisfies~~ $\in L_2^0(P_\eta)$ satisfies

$$\frac{\partial}{\partial t} \chi(\eta_t)\Big|_{t=0} = \int \chi^{(1)}_\eta \, \dot{g}_\eta \, dP_\eta \quad \text{for any such path \& corresponding } \dot{g}_\eta.$$

(Actually all you need is paths for which $\int \left(\frac{\sqrt{p_{\eta_t}} - \sqrt{p}}{t} - \frac{1}{2} \dot{g}_\eta \sqrt{p_\eta} \right)^2 d\mu \to 0$

$\Rightarrow \dot{g}_\eta \in L_2^0(P_\eta)$)

We now want carry forward this routine to a second order.

Second order Frechet Representation,

$$\chi(\eta) = \chi(\hat{\eta}) + \chi'_{\hat{\eta}}(\eta - \hat{\eta}) + \frac{1}{2}\cdot \chi''_{\hat{\eta}}(\eta - \hat{\eta}, \eta - \hat{\eta}) + O_P(\|\eta - \hat{\eta}\|^3)$$

2nd order Von Mises Representation:

$$\chi''_{\hat{\eta}}(\eta - \hat{\eta}, \eta - \hat{\eta}) = \iint \chi^{(2)}_{\hat{\eta}}(x_1, x_2)\, dP_\eta(x_1)\, dP_\eta(x_2)$$

$\to$ estimate by $U_n \chi^{(2)}_{\hat{\eta}}$ (will come back to what we mean by this later)

<u>First</u>: what does this mean in terms of 1-dimensional differentiable submodels $\{P_{\eta_t} : t \in (-1, 1)\} \subseteq \mathcal{P}$ at P_η?

Von-Mises Expansion implies

$$\chi(\eta_t) \approx \chi(\eta) + \int \chi^{(1)}_\eta(x_1)\, p_{\eta_t}(x_1)\, d\mu(x_1) + \frac{1}{2}\iint \chi^{(2)}_\eta(x_1, x_2)\, p_{\eta_t}(x_1)\, p_{\eta_t}(x_2)\, d\mu(x_1)\, d\mu(x_2)$$

$$= \chi(\eta) + \iint \chi^{(1)}_\eta(x_1)\, p_{\eta_t}(x_1)\, p_{\eta_t}(x_2)\, d\mu(x_1)\, d\mu(x_2)$$

$$+ \frac{1}{2}\iint \chi^{(2)}_\eta(x_1, x_2)\, p_{\eta_t}(x_1)\, p_{\eta_t}(x_2)\, d\mu(x_1)\, d\mu(x_2)$$

$$= \chi(\eta) + t\iint \left(\chi^{(1)}_\eta(x_1) + \frac{1}{2}\chi^{(2)}_\eta(x_1, x_2)\right) \frac{\frac{\partial}{\partial t}\prod_{i=1}^{2} p_{\eta_t}(x_i)}{\prod_{i=1}^{2} p_\eta(x_i)}\, dP_\eta(x_1)\, dP_\eta(x_2)$$

$$+ \frac{t^2}{2}\iint \left(\chi^{(1)}_\eta(x_1) + \frac{1}{2}\chi^{(2)}_\eta(x_1, x_2)\right) \frac{\frac{\partial^2}{\partial t^2}\prod_{i=1}^{2} p_{\eta_t}(x_i)}{\prod_{i=1}^{2} p_\eta(x_i)}\, dP_\eta(x_1)\, dP_\eta(x_2)$$

⇒ If we define a tangent ~~set of~~ order 2 as the set of all derivatives

$$\dot{g}_\eta(x_1,x_2) = \frac{\partial}{\partial t}\prod_{i=1}^{2} p_{\eta_t}(x_i)\Big|_{t=0} \Big/ \prod_{i=1}^{2} p_\eta(x_i)$$

$$\ddot{g}_\eta(x_1,x_2) = \frac{\partial^2}{\partial t^2}\prod_{i=1}^{2} p_{\eta_t}(x_i)\Big|_{t=0} \Big/ \prod_{i=1}^{2} p_\eta(x_i)$$

which arise from differentiable submodels $(P_{\eta_t} : t \in (-1,1))$, then $x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)}$ satisfies

$$\frac{d}{dt}x(\eta_t)\Big|_{t=0} = \frac{d}{dt}P_{\eta_t}^2\left(x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)}\right) = P_\eta^2\left(x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)}\right)\dot{g}_\eta$$

$$\frac{d^2}{dt^2}x(\eta_t)\Big|_{t=0} = \frac{d^2}{dt^2}P_{\eta_t}^2\left(x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)}\right)$$

$$= P_\eta^2\left(x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)}\right)\ddot{g}_\eta$$

Going out in this direction, we can think about going on doing such an expansion & provided we have Von-Mises ~~derivatives~~ representation of the derivatives we would expect for $j=1,\dots,m$

$$\frac{d^j}{dt^j}x(\eta_t)\Big|_{t=0} = \frac{d^j}{dt^j}P_{\eta_t}^m\left(x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)} + \cdots \frac{1}{m!}x_\eta^{(m)}\right)$$

$$= P_\eta^m\left(x_\eta^{(1)} + \frac{1}{2}x_\eta^{(2)} + \cdots \frac{1}{m!}x_\eta^{(m)}\right)\overset{\cdots}{g}_\eta \quad \leftarrow (j \text{ dots})$$

$$\uparrow \quad \frac{\frac{\partial^j}{\partial t^j}\prod_{i=1}^{m} p_{\eta_t}(x_i)\Big|_{t=0}}{\prod_{i=1}^{m} p_\eta(x_i)}$$

Also whenever $\{\dot{g}_\eta, \ddot{g}_\eta, \dots \overset{\cdots\cdot}{g}_\eta\} = L_2^0(P_\eta^m)$ (which happens for models restricted by smoothness/qualitative assumptions)

$x_\eta^{(j)}(x_1,\dots,x_j)$ is the first order IF of $x_\eta^{(j-1)}(x_1,\dots,x_{j-1})$ at x_j i.e

$$\frac{\partial}{\partial t}x_{\eta_t}^{(j-1)}(x_1,\dots,x_{j-1}) = \int x_\eta^{(j)}(x_1,\dots x_{j-1}, x_j)\,\dot{g}_\eta(x_j)\,dP_\eta(x_j)$$

this will allow us to have an estimator

$$\hat{x}_n = x(\hat{\eta}) + U_n\left(x^{(1)}_{\hat{\eta}} + \frac{1}{2} x^{(2)}_{\hat{\eta}} + \cdots \frac{1}{m!} x^{(m)}_{\hat{\eta}}\right)$$

having error of the order $O_P(\|\eta - \hat{\eta}\|^{m+1})$

$+ O_P(1/\sqrt{n})$

Unfortunately there is no free lunch. We will try to understanding through $X_1 \ldots X_n \sim \eta$ on $[0,1]$ & $x(\eta) = \int \eta^2 d\mu$.

In this case the m^{th} order tangent set ~~at~~ at η is $L_2^0(\eta^m)$ (not too difficult to verify) ~~by fact~~

We will consider ~~=~~ $m=2$ to begin with ~~and~~ and try to argue heuristically that ~~a~~ even a second order Von Mises expansion fails. Suppose not. Last time we argued that,

$x^{(1)}_\eta(x_1) = 2(\eta(x) - x(\eta))$

this was used for centering
actually $x^{(1)}_\eta = 2\eta$ works.

So ~~the~~ finding $x^{(2)}_\eta(x_1, x_2)$ relies on finding the first order ~~the~~ influence function of $\eta \to 2\eta(x_1)$ at x_2. This implies that

$$\frac{\partial}{\partial t}\eta_t(x_1)\Big|_{t=0} = \int x^{(2)}_\eta(x_1, x_2)\, \underbrace{\dot{g}_\eta(x_2)}_{\eta(x_2)}\, d\mu(x_2)$$

$$\left\| \frac{\partial}{\partial t}\eta_t(x_2)\Big|_{t=0} \over \eta(x_2)\right.$$

$$\Rightarrow \dot{g}_\eta(x_1)\eta(x_1) = \int x^{(2)}_\eta(x_1, x_2)\, \dot{g}_\eta(x_2)\eta(x_2)\, d\mu(x_2) \quad \text{for } \forall \dot{g}_\eta$$

~~$\dot{g}_\eta \in L_2$~~ $\dot{g}_\eta \in L_2(\eta)$ $\quad \dot{g}_\eta\sqrt{\eta} \in L_2(\mu)$ $\quad \dot{g}_\eta \eta \in L_1(\mu)$

$\sqrt{\eta} \in L_2(\mu)$

$\mathring{g}_\eta \in L_2(\eta)$

$\mathring{g}_\eta \to \mathring{g}_\eta(x_1)\,\eta(x_1)$

$L_2(\eta) \to \mathbb{R}$

can be written as, $\int x_\eta^{(2)}(x_1,x_2)\,\mathring{g}_\eta(x_2)\,dP_\eta(x_2)$

$$\langle x_\eta^{(2)}(x_1,\cdot)\,\mathring{g}_\eta\rangle_{P_\eta} \leftarrow \text{bounded linear functional over } L_2(P_\eta)$$

However $\mathring{g}_\eta \to \mathring{g}_\eta(x_1)\,\eta(x_1)$ is not even continuous.

so such a representation does not exist. Do not seek such a representation for all $\mathring{g}_\eta$. Indeed $\mathring{g}_\eta$ if $L \subset L_2(\mu)$ is finite dimensional, $\exists$ $\chi_L(x_1,x_2)$ s.t. $f(x_1) = \int \chi_L(x_1,x_2)\,f(x_2)\,d\mu(x_2)$ for all $f \in L$.

so if $\mathring{g}_\eta\,\eta \in L$ then

$$\mathring{g}_\eta(x_1)\,\eta(x_1) = \int \chi_L(x_1,x_2)\,\mathring{g}_\eta(x_2)\,\eta(x_2)\,d\mu(x_2)$$

$\Rightarrow \chi_L(x_1,x_2)$ works on L.

Let $e_1, e_2, \ldots$ be an o.n.b of $L_2(\mu)$ Let

$$\Pi(x_1,x_2) = \sum_{\ell=1}^{k} e_\ell(x_1)\,e_\ell(x_2)$$

$\Rightarrow$ For all $f \in \operatorname{lin}\{e_1,\ldots,e_k\}$, $f(x_1) = \int \Pi(x_1,x_2)\,f(x_2)\,d\mu(x_2)$

Let $\dot{x}_{\eta_*}^{(2)}(x_1,x_2) = 2\,\Pi(x_1,x_2)$

$$\dot{x}_\eta^{(2)}(x_1,x_2) = 2\Pi(x_1,x_2) - 2\Pi p(x_1) - 2\,\Pi p(x_2) + 2\int(\Pi p)^2$$

$$\hat{\chi}_n = \chi(\hat{\eta}) + \cancel{P}_n \chi^{(1)}_{\hat{\eta}} + \frac{1}{2} U_n \chi^{(2)}_{\hat{\eta}}$$

$$= \chi(\hat{\eta}) + \frac{1}{n} \sum_{i=1}^{n} 2\left(\hat{\eta}(X_i) - \chi(\hat{\eta})\right)$$

$$+ \frac{1}{2} \frac{1}{n(n-1)} \sum_{i \neq j} 2\Big[\Pi(X_i, X_j) - 2\int \Pi(X_i, x)\hat{\eta}(x)\, d\mu(x)$$

$$- 2\int \Pi(X_j, x)\hat{\eta}(x)\, d\mu(x)$$

$$+ 2\int (\Pi\hat{\eta})^2 d\mu(x)\Big] \quad *$$

$$\cancel{= \chi(\hat{\eta}) + \frac{1}{n} \sum_{i=1}^{n} 2(\hat{\eta}(X_i)}$$

$$\int \Pi(x_1, x_2)\, \hat{\eta}(x_2)\, d\mu(x_2)$$

$$= \int \sum_{l=1}^{k} e_l(x_1) e_l(x_2)\, \hat{\eta}(x_2)\, d\mu(x_2)$$

$$= \sum_{l=1}^{k} e_l(x_1) \int e_l(x_2)\, \hat{\eta}(x_2)\, d\mu(x_2)$$

$$\hat{\eta} = \sum_{l=1}^{k} e_l(x) \left(\frac{1}{n} \sum_{i=1}^{n} e_l(X_i)\right)$$

$$\Rightarrow \hat{\eta} \in \text{lin}\{e_1, \ldots, e_l\}$$

$$(*) \quad \hat{\chi}_n = \frac{1}{n} \sum_{i \neq j}^{n} \Pi(X_i, X_j) = \frac{1}{n} \sum_{i \neq j} \left(\sum_{l=1}^{k} e_l(X_i) e_l(X_j)\right)$$

we shall no study properties of this estimator.