Lecture 5

This lecture will be devoted to understanding a particular basis of $L_2(\mu)$, where μ is the Lebesgue measure on $([0,1], \mathcal{B}([0,1]))$, and its corresponding approximation properties for Holder-α smooth functions. For the purpose of this lecture we will denote $L_2(\mu) = L_2[0,1] = \{f : [0,1] \to \mathbb{R} : \int f^2(x)dx < \infty\}$.

The Haar Basis of $L_2[0,1]$.

$L_2[0,1]$ is a Hilbert space w.r.t the usual inner product $\langle f, g\rangle = \int f(x)g(x)dx \Rightarrow$ It has an orthonormal basis. The Haar basis is one particular such o.n.b. which we shall use due its special localization type properties. Moreover, this will ~~be~~ also help us motivate ideas from more general compactly supported wavelet bases and approximation properties in Besov spaces.

We begin with a few definitions/notations.

A system of functions $\{\varphi_k, k \in \mathbb{N}\}$, $\varphi_k \in L_2[0,1]$ is called an orthonormal system (o.n.s) if $\int \varphi_k(x)\varphi_j(x)\,dx = \delta_{jk}\ \forall j,k$

An o.n.s. is called an orthonormal basis (o.n.b) in a subspace $V \subseteq L_2[0,1]$ if for any function $f \in V$ one has a representation

$$f(x) = \sum_k c_k \varphi_k(x)$$

where the coefficients c_k satisfy $\sum_k c_k^2 < \infty$

Now consider the following simple subspace V_0 of $L_2[0,1]$ of constant functions:

$$V_0 = \{ f \in L_2[0,1] : f \text{ is a constant on } [0,1] \}$$

clearly, $f \in V_0$ iff $f(x) = c\,\varphi(x)$ for some $c \in \mathbb{R}$

where $\varphi(x) = I(x \in [0,1]) = \begin{cases} 1 & \text{if } x \in (0,1] \\ 0 & \text{o.w.} \end{cases}$

$\Rightarrow \{\varphi_{00}\}$ is an o.n.b. of V_0 where $\varphi_{00} \equiv \varphi$.

Now define a new subspace of $L_2[0,1]$ as follows

$$V_1 = \{ \cancel{h(x) = f(2x) : f \in V_0} \}$$

= all functions in $L_2[0,1]$ that are constant on $(0,\frac{1}{2}]$ & $(\frac{1}{2},1]$.

Indeed, $V_0 \subset V_1$ and an o.n.b of V_1 is given by $\{\varphi_{10}, \varphi_{11}\}$ where $\varphi_{1k}(x) = \sqrt{2}\,\varphi(2x-k)$ $\quad k \in \{0,1\}$

One can reiterate this process and define in general

$$V_j = \{ \cancel{h(x) = f(2^j x) : f \in V_0} \} = \text{all function in } L_2[0,1] \text{ that are constant on } \left(\tfrac{k}{2^j}, \tfrac{k+1}{2^j}\right]\ k = 0, \ldots, 2^j - 1,$$

Then V_j is a linear subspace of functions which are constant on the intervals $\left\{ \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right], k = 0, \ldots, 2^j - 1 \right\}$ having an o.n.b.

$$\varphi_{jk}(x) = 2^{j/2}\,\varphi(2^j x - k) \qquad k = 0, 1, \ldots, 2^j - 1.$$

this function is constant $(2^{j/2})$ on $\left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]$ & hence has norm 1.

Moreover, $V_0 \subset V_1 \subset V_2 \subset \ldots$

Continuing this process one can approximate the whole $L_2[0,1]$

<u>Theorem:</u> $\bigcup_{j=0}^{\infty} V_j$ is dense in $L_2([0,1])$.

<u>Proof:</u> The proof is simple. Firstly continuous functions are dense in $L_2[0,1]$. (This can be proved by starting from simple functions by Urysons lemma, and then using dominated convergence to show simple functions are dense in $L_2[0,1]$). Finally continuous functions on $[0,1]$ can be approximated in $L_2[0,1]$ with functions constant on dyadic intervals. ~~Stated in a simpler way, if $f \in L_2[0,1]$ is nonnegative, then $\exists f_n \in \bigcup_{j=0}^{\infty} V_j$ s.t. $f_n \uparrow f$ pointwise. Thereafter dominated convergence proves the result. For general f write $f = f^+ - f^-$ & complete.~~ ∎

However $\bigcup_{j=0}^{\infty} V_j$ is not a basis of $L_2[0,1]$ since for example $\varphi_{00}(x) = (\varphi_{10}(x) + \varphi_{11}(x))/\sqrt{2}$

However it can be transformed into an o.n.b. through orthogonalization as follows.

Since $V_0 \subset V_1$, let W_0 be the orthogonal complement of V_0 in V_1 i.e. $V_1 = W_0 \oplus V_0$ i.e. any $v_1 \in V_1$ can be uniquely represented as $v_1 = v_0 + w_0$ with $v_0 \in V_0$ & $w_0 \in W_0$ where $v_0 \perp w_0$.

Now the question becomes that whether we can adequately describe the space W_0. It turns out its easy to produce an o.n.b. for W_0

Let $\psi(x) = \psi_{00}(x) = \begin{cases} -1 & x \in [0, \frac{1}{2}] \\ 1 & x \in (\frac{1}{2}, 1] \end{cases}$

Then $\{\psi_{00}\}$ ~~is an a~~ generates W_0. To see this it is enough to show the following:

(i) ψ_{00} is orthogonal to $V_0 \rightarrow$ this is trivial since ψ_{00} is orthogonal to $\varphi(x)$.

(ii) Every $f \in V_1$ has a unique representation in terms of $\{\varphi_{00}, \psi_{00}\}$. To see this let $f \in V_1$. Then $\exists\, c_0$ and c_1 s.t.

$$f(x) = c_0 \varphi_{10}(x) + c_1 \varphi_{11}(x)$$

Therefore its enough to show that φ_{10} & φ_{11} are linear combination of φ_{00} and ψ_{00}.

$$\varphi_{10}(x) = \sqrt{2}\,\varphi(2x) = \sqrt{2}\, I\left(x \in (0, \tfrac{1}{2}]\right) = \sqrt{2}\left\{\varphi_{00}(x) - \psi_{00}(x)\right\}/2$$
$$= \frac{1}{\sqrt{2}}\left\{\varphi_{00}(x) - \psi_{00}(x)\right\}$$

Similarly, $\varphi_{11}(x) = \sqrt{2}\,\varphi(2x-1) = \sqrt{2}\, I\left(x \in (\tfrac{1}{2}, 1]\right)$
$$= \sqrt{2}\left\{\varphi_{00}(x) + \psi_{00}(x)\right\}/\sqrt{2}$$
$$= \frac{1}{\sqrt{2}}\left\{\varphi_{00}(x) + \psi_{00}(x)\right\}$$

Therefore $V_1 = V_0 \oplus W_0 \Rightarrow \{\varphi_{00}, \psi_{00}\}$ is an o.n.b. of V_1.

Following the same idea, one can extend the construction to every V_j, $j \geq 0$ to obtain

$$V_{j+1} = V_j \oplus W_j$$

where W_j is the ortho-complement of V_j in V_{j+1}. Similar to the proof above, one can check that

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^{j/2}x - k), \quad k = 0, 1, \ldots, 2^j - 1 \text{ is an}$$

o.n.b. of W_j.

Formally, we can write

$$V_{j+1} = V_j \oplus W_j = V_{j-1} \oplus W_{j-1} \oplus W_j = \dots = V_0 \oplus W_0 \oplus \dots \oplus W_j = V_0 \oplus \bigcup_{l=0}^{j} W_l$$

We already know that $\bigcup_{j=0}^{\infty} V_j$ is dense in $L_2[0,1]$ i.e.

$$\overline{\bigcup_{j=0}^{\infty} V_j} = L_2[0,1] \quad \leftarrow \text{closure in } L_2[0,1]$$

Using orthogonal sum decomposition of V_j we write,

$$L_2[0,1] = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j \qquad (*)$$

where $(*)$ is the symbolic representation of the fact that every $f \in L_2[0,1]$ can be represented as

$$f(x) = \alpha_{00}\,\varphi_{00}(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{jk}\,\psi_{jk}(x) \qquad (**)$$

$\alpha_{00} = \langle f, \varphi_{00} \rangle$, $\beta_{jk} = \langle f, \psi_{jk} \rangle$. $\alpha_{00} \leftarrow$ scaling coeff, β_{jk}'s $\leftarrow$ detail coefficients.

where the infinite sum is taken to converge in $L_2[0,1]$.

<u>Theorem</u>: $\{\varphi_0, \psi_{jk}, k = 0, \dots, 2^j-1, j = 0, 1, \dots\}$ is an o.n.b. of $L_2[0,1]$. (This is called the Haar Wavelet Basis of $L_2[0,1]$)

<u>Remark</u>: $(**)$ is a special example of a "wavelet expansion" $\rightarrow$ one which corresponds to our special choice of φ and ψ above. The function φ is called the father wavelet & ψ the mother wavelet. (ψ_{jk}'s are called the children).

Also the mother wavelet ψ may be defined in a different way e.g. $\psi(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ -1 & x \in (1/2, 1] \end{cases}$ $\rightarrow$ In fact given ~~given~~ a father there can be several mother wavelets. ∎

Now that we have an o.n.b. of $L_2[0,1]$ we want to study properties of the projection kernel $\Pi(x_1, x_2)$ (introduced last time) corresponding to this basis while approximating elements in $H(\alpha, M) = \left\{ f : [0,1] \to \mathbb{R} : \sup_{x,y \in [0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq M \right\}$ for $0 < \alpha < 1$ and $M > 0$.

To study $\Pi(x_1, x_2)$ we need to fix a lexicographic ordering of the Haar Basis introduced above. The basis can be written in the following order

$$\underbrace{\varphi_{00}}_{V_0},\ \underbrace{\psi_{00}}_{W_0},\ \underbrace{\psi_{10}, \psi_{11}}_{W_1},\ \ldots\ ,\ \underbrace{\psi_{j0}, \ldots \psi_{j\,2^j-1}}_{W_j} \ldots$$

at jth level of approximation, Then, we create $\Pi(x_1, x_2)$ by including all the basis functions from $V_0, W_0, W_1, \ldots, W_{j-1}$ $\Rightarrow$ we have a total of

$$1 + \sum_{l=0}^{j-1} 2^l = 1 + 2^j - 1 = 2^j \text{ basis functions.}$$

In terms of our previous lecture $k = 2^j$. and $\Pi(x_1, x_2)$ can be written as

$$\Pi(x_1, x_2) := K_{V_j}(x_1, x_2) = \varphi_{00}(x_1)\varphi_{00}(x_2) + \sum_{l=0}^{j-1} \sum_{k=0}^{2^l-1} \psi_{lk}(x_1)\psi_{lk}(x_2)$$

Then $K_{V_j}(x_1, x_2)$ is the projection kernel onto V_j. It can also be shown by direct computation that

$$K_{V_j}(x_1, x_2) = \sum_{k=0}^{2^j-1} \varphi_{jk}(x_1)\,\varphi_{jk}(x_2) \quad \text{where recall that}$$

$\varphi_{jk}(x) = 2^{j/2}\varphi(2^j x - k)$ $k = 0, 1, \ldots, 2^j - 1$ is an o.n.b. of V_j & therefore K_{V_j} is a projection kernel onto V_j.

This representation is particularly useful for the purpose of showing projection properties ~~of~~ onto V_j.

Recall that we want to show that when $f \in H(\alpha, M)$ then $\exists$ $C(M)$ s.t.

$$\| \Pi_{V_j} f - f \|_\infty \le k^{-\alpha} = 2^{-j\alpha}$$ since $k = 2^j$ in this case.

This is collected in the next theorem.

__Theorem__: Suppose $0 < \alpha < 1$. Then ~~$\exists$ a constant $C(M)$~~ (depending ~~on M) s.t.~~

$$\sup_{f \in H(\alpha, M)} \| \Pi_{V_j} f - f \|_\infty \le 2^{-j\alpha} M$$

__Proof__: Note that

$$K_{V_j}(x_1, x_2) = \sum_{k=0}^{2^j - 1} \varphi_{jk}(x_1) \varphi_{jk}(x_2)$$

$$= 2^j \sum_{k=0}^{2^j - 1} I\left(x_1 \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) I\left(x_2 \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right)$$

$$\Rightarrow \Pi_{V_j} f(x_1) = \int K_{V_j}(x_1, x_2) f(x_2)\, dx_2$$

$$= 2^j \sum_{k=0}^{2^j - 1} \left(\int_{k/2^j}^{(k+1)/2^j} f(x)\, dx \right) I\left(x_1 \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right)$$

Also,

$$f(x) = 2^j \int_{k/2^j}^{(k+1)/2^j} f(x)\, dx' \qquad (*)$$

- Therefore to bound the supremum norm $\|\Pi_{V_j} f - f\|_\infty$ take any $x \in [0,1]$. Then $\exists$ a unique $k \in \{0, 1, \dots, 2^j - 1\}$ s.t. $x \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right] \Rightarrow \Pi_{V_j} f(x) = 2^j \int_{k/2^j}^{k+1/2^j} f(x')\,dx'$

$$\Rightarrow \left|\Pi_{V_j} f(x) - f(x)\right| = \left| 2^j \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} f(x')\,dx' - f(x)\right|$$

$$= \left| 2^j \int_{\frac{k}{2^j}}^{(k+1)/2^j} f(x')\,dx' - 2^j \int_{\frac{k}{2^j}}^{(k+1)/2^j} f(x)\,dx'\right| \quad \text{(from (*))}$$

$$\leq 2^j \int_{\frac{k}{2^j}}^{(k+1)/2^j} |f(x) - f(x')|\,dx' \leq 2^j M \int_{\frac{k}{2^j}}^{(k+1)/2^j} |x - x'|^\alpha\,dx' \quad \text{(by def. of } H(\alpha, M))$$

$$\leq 2^j \int_{k/2^j}^{k+1/2^j} M\, 2^{-j\alpha}\,dx' \quad \text{as } |x - x'| \leq 2^{-j} \text{ since } x \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]$$

$$= M 2^{-j\alpha} \quad \text{as required} \quad \blacksquare$$

Remark: From the above calculation it is also clear where we needed $0 < \alpha < 1$. For $\alpha > 1$, the definition of Holder α-smooth functions are as follows. Let $\lfloor \alpha \rfloor$ be the ~~smallest~~ largest integer strictly smaller than. Then

$$H(\alpha, M) = \left\{ f : [0,1] \to \mathbb{R} : \sup_{x, y \in [0,1]} \frac{\left|f^{(\lfloor\alpha\rfloor)}(x) - f^{(\lfloor\alpha\rfloor)}(y)\right|}{|x - y|^\alpha} \leq M \right\}$$

Indeed the above proof does not work for $\alpha > 1$ since Haar basis cannot approximate smooth functions well (its approximations are always step functions).

Note that if $\sup_{x, y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq M$ for $\alpha > 1 \Rightarrow f$ is constant function.

Remark: Haar basis can consistently approximate $f \in H(\alpha, M)$ for $\alpha > 1$ still, but fails to achieve the desired $2^{-j\alpha}$ rate of convergence.

Theorem: Let $f : [0,1] \to \mathbb{R}$ be measurable.

(i) If f is bounded on $[0,1]$ and continuous at $x \in [0,1]$ then $K_{V_j} f(x) \to f(x)$ as $j \to \infty$

(ii) If f is continuous on $[0,1]$; then $\| K_{V_j} f - f \|_\infty \to 0$ as $j \to \infty$

(iii) If $f \in L^p[0,1]$ for some $1 \le p < \infty$ then $\| K_{V_j} f - f \|_p \to 0$ as $j \to \infty$

Proof: Mimic proof of Proposition 4.1.1 of Giné & Nickl (2015) (Mathematical Foundations of Infinite Dimensional Statistical Models)