# Lecture 6

In this lecture we will explore lower bound arguments to supplement our upper bound results for estimating $\int \eta^2 d\mu$ for $\eta \in \mathcal{P}(\alpha, M, B) = \left\{\eta : \int \eta\, d\mu = 1, \eta \geq 0, \|\eta\|_\infty \leq B, \sup_{x,y} \frac{|\eta(x) - \eta(y)|}{|x-y|^\alpha} \leq M\right\}$ for $0<\alpha<1$ and given $0 < M, B < \infty$.

We demonstrated estimators such that

$$\sup_{\eta \in \mathcal{P}(\alpha, M, B)} \mathbb{E}_\eta \left(\hat{x}_n - x(\eta)\right)^2 \leq C(M,B)\, n^{-\frac{8\alpha}{4\alpha+1}} \quad \text{for } \alpha \leq 1/4$$

$$\leq C(M,B)\, n^{-1} \quad \text{for } \alpha > 1/4$$

where $C(M,B)$ denotes a constant/number depending on $M, B$.

We now want to show that we cannot improve on this in the following sense.

Let us focus on the case $0<\alpha<1/4$ (The other case is technically simpler)

We want to show that for any estimator $\hat{x}_n$

$$\sup_{\eta \in \mathcal{P}(\alpha, M, B)} \mathbb{E}_\eta \left(\hat{x}_n - x(\eta)\right)^2 \geq c'(M,B)\, n^{-\frac{8\alpha}{4\alpha+1}}$$

for some number $c'(M,B)$ depending on $(M,B)$.

## General Idea of Proving Such a Lower Bound

Lower bounds ~~arguments~~ are often proved by testing arguments as follows. ~~If $P_n$ and $Q_n$ are ...~~

~~$\{P^n_\eta : \eta \in \mathcal{P}(\alpha, M, B)\}, x(\eta$~~

Suppose you ~~want to estimate~~ have a sample $X_1, X_2, \ldots, X_n$ from a ~~density~~ distribution $P \in \mathcal{P}$ and we wish to estimate $\psi(P) : \mathcal{P} \to \mathbb{R}$ and are interested in understanding

$$(*) \qquad \inf_{T_n} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left(T_n - x(P)\right)^2 \qquad T_n := T_n(X_1, \ldots, X_n) \text{ any measurable map.}$$

If $P_n$ and $Q_n$ are in convex hulls of the sets $\{P^n : P \in \mathcal{P}, \Psi(P) \le 0\}$ and $\{P^n : P \in \mathcal{P}, \Psi(P) \ge \varepsilon_n\}$ for a sequence $\varepsilon_n$, such that there exists no sequence of tests between $P_n$ and $Q_n$ with both ~~to~~ probabilities of type I & type II error tending to 0, then (*) is at least "as large as" $\varepsilon_n^2$. We make this more formal below.

Let $\mathcal{P}$ be a set of probability measures on a measurable space $(\Omega, \mathcal{A})$ and let $\Psi : \mathcal{P} \to \mathbb{R}$ be a functional. Let $P^n$ be the $n$-fold product of $P$. Define

$$\mathcal{P}_{n,\le 0} := \{P^n : P \in \mathcal{P}, \Psi(P) \le 0\}, \quad \mathcal{P}_{n,\ge \varepsilon} := \{P^n : P \in \mathcal{P}, \Psi(P) \ge \varepsilon\}$$

We are given $n$ iid observations $X_1, \dots, X_n$ from $P \in \mathcal{P}$. In the following, an estimator is a measurable map $T_n : (\Omega^n, \mathcal{A}^n) \to \mathbb{R}$, and a test $\varphi_n$ is a measurable map taking values in $\{0,1\}$.

__Theorem 1:__ Suppose that for some $\varepsilon_n \to 0$ there exists measures $P_n$ and $Q_n$ in the convex hulls of $\mathcal{P}_{n,\le 0}$ and $\mathcal{P}_{n,\ge \varepsilon_n}$ respectively, such that

$$\liminf_{n\to\infty} \left(P_n \varphi_n + Q_n(1-\varphi_n)\right) > 0$$

for any sequence of tests $\varphi_n$. Then

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n^2} \inf_{T_n} \sup_{P \in \mathcal{P}} \mathbb{E}_P\left(T_n - \Psi(P)\right)^2 > 0$$

Proof: First note that it is equivalent to prove that for any sequence of estimators $T_n$,

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n^2} \sup_{P\in\mathcal{P}} \mathbb{E}_P (T_n - \Psi(P))^2 > 0$$

We will actually prove that

$$\liminf_{n\to\infty} \sup_{P\in\mathcal{P}} P\left(|T_n - \Psi(P)| \geq \varepsilon_n/2\right) > 0 \qquad (*)$$

$\Rightarrow$ the result by Chebychev's inequality

The proof follows by contradiction. If a $T_n$ satisfies $(*)$ then one can construct a test $\varphi_n$ between $\mathcal{P}_{n,\leq 0}$ & $\mathcal{P}_{n,\geq \varepsilon_n}$ using this estimator $T_n$.

More concretely takes $\varphi_n = I(T_n \geq \varepsilon_n/2)$

We now show that $\liminf_{n\to\infty} (P_n\varphi_n + Q_n(1-\varphi_n)) = 0$ for all $P_n \in \text{conv}(\mathcal{P}_{n,\leq 0})$ and $Q_n \in \text{conv}(\mathcal{P}_{n,\geq\varepsilon_n})$

$P_n\varphi_n = \mathbb{E}_{P_n}\varphi_n = P_n(\varphi_n = 1) = P_n(T_n \geq \varepsilon_n/2)$

Now for every $P^n \in \mathcal{P}_{n,\leq 0}$, $\Psi(P) \leq 0 \Rightarrow P^n\varphi_n \leq P(|T_n - \Psi(P)| \geq \varepsilon_n/2) \longrightarrow 0$

$\Rightarrow$ for any $P_n \in \text{conv}(\mathcal{P}_{n,\leq 0})$, $P_n\varphi_n \longrightarrow 0$

Also for every $P^n \in \mathcal{P}_{n,\geq\varepsilon_n}$, $\Psi(P) \geq \varepsilon_n \Rightarrow P^n(1-\varphi_n) \leq P(|T_n - \Psi(P)| \geq \varepsilon_n/2) \longrightarrow 0$

$\Rightarrow$ for any $Q_n \in \text{conv}(\mathcal{P}_{n,\geq\varepsilon_n})$, $Q_n(1-\varphi_n) \longrightarrow 0$

$\Rightarrow$ For any $P_n \in \text{conv}(\mathcal{P}_{n,\leq 0})$ and $Q_n \in \text{conv}(\mathcal{P}_{n,\geq\varepsilon_n})$

$$\liminf_{n\to\infty} (P_n\varphi_n + Q_n(1-\varphi_n)) = 0 \longrightarrow \text{a contradiction.}$$

Therefore the question now becomes to find conditions between $P_n$ and $Q_n$ such that one cannot distinguish between $P_n$ and $Q_n$. In the following assume $Q_n \ll P_n$ so that $dQ_n/dP_n$ is well defined.

Then for any test $\varphi_n$,

$$(**) \quad P_n\varphi_n + Q_n(1-\varphi_n) = P_n\varphi_n + P_n(1-\varphi_n)\frac{dQ_n}{dP_n}$$

$$= P_n\left[\varphi_n + (1-\varphi_n)\frac{dQ_n}{dP_n}\right]$$

$$\geq P_n\left[\varphi_n + (1-\eta)(1-\varphi_n)\, I\left(\frac{dQ_n}{dP_n} \geq 1-\eta\right)\right]$$

$$\geq (1-\eta)\, P_n\left(\frac{dQ_n}{dP_n} \geq 1-\eta\right)$$

$$= \cancel{(1-\eta)\left(1 - P_n\left(\frac{dQ_n}{dP_n} \leq 1-\eta\right)\right)}$$

$$= \cancel{(1-\eta)\, P_n\left(\frac{dQ_n}{dP_n} \geq 1-\eta\right)}$$

$$= (1-\eta)\left(1 - P_n\left(\frac{dQ_n}{dP_n} \leq 1-\eta\right)\right)$$

$$= (1-\eta)\left(1 - P_n\left(1-\frac{dQ_n}{dP_n} \geq \eta\right)\right)$$

$$\geq (1-\eta)\left(1 - P_n\left(\left|\frac{dQ_n}{dP_n} - 1\right| \geq \eta\right)\right)$$

$$\geq (1-\eta)\left(1 - \frac{E_{P_n}\left|\frac{dQ_n}{dP_n} - 1\right|}{\eta}\right)$$

$$\geq (1-\eta)\left(1 - \frac{\sqrt{E_{P_n}\left(\frac{dQ_n}{dP_n} - 1\right)^2}}{\eta}\right)$$

(Cauchy Schwarz inequality with $E_{P_n}\frac{dQ_n}{dP_n} = 1$)

$$= (1-\eta)\left[1 - \frac{1}{\eta} \times \sqrt{E_{P_n}\left(\frac{dQ_n}{dP_n}\right)^2 - 1}\,\right]$$

If $E_{P_n}\left(\frac{dQ_n}{dP_n}\right)^2 = 1 + o(1) \Rightarrow P_n\varphi_n + Q_n(1-\varphi_n) \geq (1-\eta)(1-o(1))$

(Actually enough to show $E_{P_n}\left(\frac{dQ_n}{dP_n}\right)^2$ bounded)

According to the above calculations, the strategy to prove a lower bound on the estimation rate of $\Psi(P)$ for $P \in \mathcal{P}$ is now clear.

strategy to prove a lower bound of $\varepsilon_n^2$

(1) Find probability measures $P_{10}^n, \dots, P_{m_n 0}^n \in \mathcal{P}_{n, \leq 0}$ and $P_{11}^n, \dots, P_{m_n 1}^n \in \mathcal{P}_{n, \geq \varepsilon_n}$.

(2) Take one particular convex combination of these (usually equal weight) and call them $P_n$ and $Q_n$.

(3) compute $E_{P_n}\left(\frac{dQ_n}{dP_n}\right)^2$ to show it ~~behaves like 1+o(1)~~ stays bounded.

Definition: Let $P, Q$ be probability measures on a measurable space $(\Omega, \mathcal{A})$ such that $Q \ll P$. Then the $\chi^2$-divergence between $Q$ and $P$ is

$$\chi^2(Q, P) = \mathbb{E}_P\left(\frac{dQ}{dP} - 1\right)^2$$

Remark: In step (3) above we therefore want to show that $\chi^2(Q_n, P_n) \to 0$ as $n \to \infty$. This hopefully makes sense, since $\chi^2(Q_n, P_n)$ measures some kind of distance between $Q_n$ and $P_n$, and if these probability measures are "too close to each other" you cannot hope to successfully distinguish between them. This is made concrete by the series of steps (**) on page 4.

<u>Remark</u>: A closer look at theorem 1 actually shows the following. ~~[illegible]~~ If one can construct two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of $P^n$ from $P \in \mathcal{P}$ such that $\inf_{P \in \mathcal{P}_2} \Psi(P) - \sup_{P \in \mathcal{P}_1} \Psi(P) \geq \varepsilon_n$ but there exits $P_n$ and $Q_n$ in the convex hulls of $\mathcal{P}_1$ and $\mathcal{P}_2$ such that you cannot successfully test between them, then the minimax rate of estimation of $\Psi(P)$ over $\mathcal{P}$ is at least $O(\varepsilon_n^2)$.

We are now ready to prove an estimation lower bound for estimating $\chi(\eta) = \int \eta^2 d\mu$ over $\mathcal{P}(\alpha, M, B)$

## Lower Bound for Estimating Quadratic Functionals

We closely follow the straligy laid down above. For the set $\mathcal{P}_1$ (null hypothesis) we simply take the uniform distribution i.e. $\mathcal{P} = \{P^n: P = U[0,1]\}$. Therefore any convex combination is again the n-fold product of the uniform. Also, the value of the function on $\mathcal{P}_1$ is $\chi(\eta) = 1$.

Therefore if we want to demonstrate a lower bound of $O(\varepsilon_n^2)$ we want to create $\mathcal{P}_2 = \{P^n : |\chi(\eta) - 1| \geq \varepsilon_n^2,\ \eta = \frac{dP}{d\mu}\}$

For $0 < \alpha < \frac{1}{4}$, we want $\varepsilon_n^2 = n^{-\frac{8\alpha}{4\alpha+1}} \Rightarrow \varepsilon_n = n^{-\frac{4\alpha}{4\alpha+1}}$

Now we need to find elements from $\mathcal{P}_2$ and take their convex combinations to create a $Q_n$. Then we need to bound $\chi^2(Q_n, P^n)$ where $P = U[0,1]$.

Creating $Q_n$: The idea behind creating $Q_n$ is the following. You want to find as many distributions as possible (because more distributions you find to create your convex combination, the more difficult you make your testing problem, intuitively) such that its difficult to test against $U[0,1]$. Indeed you want to make these choices as close to $U[0,1]$ as possible, but constrained by the fact that $\int \eta^2 d\mu \geq \epsilon_n + 1$. Therefore you find distributions with $\int \eta^2 d\mu = \epsilon_n + 1$ while $\eta \in \mathcal{P}(\alpha, M, B)$. These you can create as follows.

Let $\eta_{\underset{\sim}{\lambda}}(x) = 1 + \underset{\substack{\uparrow \\ \text{(to be decided later)}}}{C}\, 2^{-j\alpha}\, 2^{-j/2} \sum_{k=0}^{2^j-1} \lambda_k \Psi_{jk}(x)$ where $\{\Psi_{jk}, k = 0, \dots, 2^j-1\}$ are the mother wavelets at resolution $j$, and $\underset{\sim}{\lambda} = (\lambda_0, \dots, \lambda_{2^j-1}) \in \{-1, +1\}^{2^j}$ (note we need to take $\Psi$ to be continuous $\Rightarrow$ Haar will not work)

So we have a collection of $2^{2^j}$ functions/distributions. In order for these to work, we need to verify the following

(i) $\eta_\lambda \in \mathcal{P}(\alpha, B, M)$ for all $\underset{\sim}{\lambda}$.

(ii) $\int \eta_\lambda^2 d\mu \geq 1 + C\epsilon_n$ for some $C > 0$.

(iii) $\chi^2\left(\frac{1}{2^{2^j}} \sum_{\underset{\sim}{\lambda}} P^n_{\eta_{\underset{\sim}{\lambda}}},\ U[0,1]^n\right)$ needs to controlled

Verification of (i): Fix $\underset{\sim}{\lambda} \in \{-1,+1\}^{2^j}$ and $x, y \in [0,1]$.

$$|\eta_\lambda(x) - \eta_\lambda(y)| = c\, 2^{-j\alpha} \cdot 2^{-j/2} \left| \sum_{k=0}^{2^j-1} 2^{j/2} (-1)^{k+1} \left\{ I\left(x \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) - I\left(y \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) \right\} \right|$$

$$= c\, 2^{-j\alpha} \left| \sum_{k=0}^{2^j-1} I\left(x \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) - I\left(y \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) \right| \quad (*)$$

Now if both $x, y \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, then $(*) = 0$

$\Rightarrow |\eta_\lambda(x) - \eta_\lambda(y)| \le M |x-y|^\alpha$

However, if $x \in \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right]$ and $y \in \left(\frac{k'}{2^j}, \frac{k'+1}{2^j}\right]$ for $k \ne k'$ then $|x-y| \geq 2^{-j}$ and

$$|\eta_\lambda(x) - \eta_\lambda(y)| \le (2c)\, 2^{-j\alpha} \le (2c)\, |x-y|^\alpha \le M |x-y|^\alpha$$

if $c \le M/2$

Also, $\|\eta_\lambda\|_\infty \le (1+\varepsilon)$ for $c\, 2^{-j\alpha} \le \varepsilon$

$\Rightarrow j \geq \frac{\log(c\varepsilon)}{\alpha \log 2}$

Our choice of $j$ will grow with $n$ so will satisfy this condition.

$\Rightarrow \eta_\lambda \in \mathcal{P}(\alpha, M, B) \ \forall \underset{\sim}{\lambda}$, $B > 1$.

Verification of (ii): Fix $\underset{\sim}{\lambda} \in \{-1,+1\}^{2^j}$.

$$\int \eta_\lambda^2(x)\,dx = 1 + c^2 2^{-2j\alpha}\, 2^{-j} \sum_{k=0}^{2^j-1} \lambda_k^2 \int \psi_{jk}^2(x)\,dx + \text{cross terms}$$

0 by orthogonality

(also $\int \psi_{jk}(x)\,dx = 0$)

$$= 1 + c^2 2^{-2j\alpha}\, 2^{-j} \sum_{k=0}^{2^j-1} 1 \times 1 = 1 + c^2 2^{-2j\alpha}$$

Need, $c^2 2^{-2j\alpha} \geq \varepsilon_n^* = n^{-\frac{4\alpha}{4\alpha+1}} \Rightarrow c\, n^{\frac{2\alpha}{4\alpha+1}} \geq 2^{j\alpha}$

$\Rightarrow 2^j \le c^{\frac{1}{\alpha}}\, n^{\frac{2}{4\alpha+1}}$

Take $2^j = \left[c^{\frac{1}{\alpha}} n^{\frac{2}{4\alpha+1}}\right] \Rightarrow \int \eta_\lambda^2 dx \geq 1+\varepsilon_n$ for all $\underset{\sim}{\lambda}$

and $\varepsilon_n = n^{-\frac{4\alpha}{4\alpha+1}}$

Verification of (iii)

We need to find the $\chi^2\left(\frac{1}{M}\sum_{\underset{\sim}{\lambda}} P^n_{\eta_\lambda}, P_0^n\right)$ and show its $o(1)$ where $M = 2^{2^j}$ and $P_0 = U[0,1]$

Recall that $\chi^2\left(\frac{1}{M}\sum_{\underset{\sim}{\lambda}} P^n_{\eta_\lambda}, P_0^n\right) = \mathbb{E}_{P_0^n}\left(\frac{d\frac{1}{M}\sum_{\underset{\sim}{\lambda}} P^n_{\eta_\lambda}}{dP_0^n}\right)^2 - 1$

$$= \mathbb{E}_{P_0^n}\left(\frac{1}{M}\sum_{\underset{\sim}{\lambda}} \frac{dP^n_{\eta_\lambda}}{dP_0^n}\right)^2 - 1$$

Therefore its enough to show $\mathbb{E}_{P_0^n}\left(\frac{1}{M}\sum_{\underset{\sim}{\lambda}} \frac{dP^n_{\eta_\lambda}}{dP_0^n}\right)^2 = 1+o(1)$

$$\mathbb{E}_{P_0^n}\left(\frac{1}{M}\sum_{\underset{\sim}{\lambda}} \frac{dP^n_{\eta_\lambda}}{dP_0^n}\right)^2 = \mathbb{E}_{P_0^n}\left[\frac{1}{M}\sum_{\underset{\sim}{\lambda}} \prod_{i=1}^n \eta_\lambda(x_i)\right]^2$$

$$= \frac{1}{M^2}\sum_{\underset{\sim}{\lambda}, \underset{\sim}{\lambda}'} \mathbb{E}_0\left(\prod_{i=1}^n \eta_\lambda(x_i) \prod_{i=1}^n \eta_{\lambda'}(x_i)\right) \quad \text{where } \mathbb{E}_0 = \mathbb{E}_{P_0^n}$$

$$= \frac{1}{M^2}\sum_{\underset{\sim}{\lambda}, \underset{\sim}{\lambda}'} \prod_{i=1}^n \mathbb{E}_0\left(\eta_\lambda(x_i)\eta_{\lambda'}(x_i)\right) \qquad (*)$$

Now $\mathbb{E}_0\left(\eta_\lambda(x_i)\eta_{\lambda'}(x_i)\right) = \int \left(1+c\,2^{-j\alpha}\cdot 2^{-j/2}\sum \lambda_k \psi_{jk}(x)\right)$

$$\left(1+c\,2^{-j\alpha}\,2^{-j/2}\sum \lambda_k' \psi_{jk}(x)\right) dx$$

$$= \left(1 + c^2\, 2^{-2j\alpha}\, 2^{-j} \sum \lambda_k \lambda_k'\right)$$

Pluggin in (*)

$$\mathbb{E}_{P_0^n}\left(\frac{1}{M}\sum_{\underset{\sim}{\lambda}} \frac{dP^n_{\eta_\lambda}}{dP_0^n}\right)^2 = \frac{1}{M^2}\sum_{\underset{\sim}{\lambda}, \underset{\sim}{\lambda}'}\left(1 + c^2\, 2^{-2j\alpha}\, 2^{-j}\sum \lambda_k \lambda_k'\right)^n$$

$$= \mathbb{E}\left[\left(1 + \underbrace{c^2\, 2^{-2j\alpha}\, 2^{-j}}_{\theta}\, Y_j\right)^n\right] \quad \text{where } Y_j = \sum_{k=1}^{2^j - 1} R_k$$

where $R_k \overset{iid}{\sim} \{-1, +1\}$ w.p. $1/2$ each

$$= \mathbb{E}\left[(1 + \theta Y_j)^n\right]$$

Now note that $e^x \geq 1 + x$ for all $x$.

Therefore $\mathbb{E}\left((1 + \theta Y_j)^n\right) \leq \mathbb{E} \exp(\theta n Y_j)$

$$= \left(\frac{\exp(n\theta) + \exp(-n\theta)}{2}\right)^{2^j}$$

$$= \{\cosh(n\theta)\}^{2^j}$$

$$n\theta = c^2\, 2^{-2j\alpha} \cdot n\, 2^{-j} = c^2 \cdot n\, 2^{-j(2\alpha+1)}$$

$$\leq c^2 \cdot n \cdot \left(\frac{c^{\frac{1}{\alpha}}\, n^{\frac{2}{4\alpha+1}}}{2}\right)^{-(2\alpha+1)} \quad \text{as } 2^j \geq \frac{c^{\frac{1}{\alpha}}\, n^{\frac{2}{4\alpha+1}}}{2} \;\; \forall n$$

$$= c^{2 - \frac{2\alpha+1}{\alpha}} \cdot 2^{-(2\alpha+1)} \cdot n^{1 - \frac{2(2\alpha+1)}{4\alpha+1}}$$

$$= c^{-\frac{1}{\alpha}} \cdot 2^{-(2\alpha+1)}\, n^{-\frac{1}{4\alpha+1}} \to 0 \quad \text{as } n \to \infty.$$

Now $\cosh(z) = 1 + z^2/2 + o(z^2)$ as $z \to 0$

$$\Rightarrow \mathbb{E}_{P_0^n}\left(\frac{1}{M}\sum_{\lambda} \frac{dP_\lambda^n}{dP_0^n}\right)^2 \leq \left(1 + \frac{n^2\theta^2}{2} + o(n^2\theta^2)\right)^{2^j}$$

$$\leq \left(1 + n^2\theta^2\right)^{2^j} \quad \text{for } \cancel{\forall n}$$

$$= \left(1 + \frac{n^2 2^j \theta^2}{2^j}\right)^{2^j} \leq \exp\left(2^j n^2 \theta^2\right)$$

$$= \exp\left(c^{-\frac{2}{\alpha}} 2^{-2(4\alpha+1)} n^{-\frac{2}{4\alpha+1}} \cdot n^{\frac{2}{4\alpha+1}}\right)$$

$$= \exp\left(c^{-2/\alpha} 2^{-2/4\alpha+1}\right)$$

This implies by calculations on page (4) of today's lecture notes,

$$P_n \varphi_n + Q_n(1-\varphi_n) \geq (1-\eta)\left[1 - \frac{1}{\eta}\left(\exp\left(c^{-2/\alpha} 2^{-2/4\alpha+1}\right) - 1\right)^{1/2}\right]$$

$$> 0 \quad \text{for suitable choice of } \eta.$$

$$\Rightarrow \liminf_{n\to\infty} \inf_{T_n} \sup_{\eta \in \mathcal{P}(\alpha, M, B)} E_\eta \left(\hat{x}_n - x(\eta)\right)^2 \Big/ n^{-\frac{8\alpha}{4\alpha+1}} > 0 \quad \text{for } 0<\alpha<\tfrac{1}{4}$$

as promised.

Exc: Verify the rate $e_n^2 = 1/n$ for $\alpha \geq 1/4$.