Differential Equations

Differential equations describe relationships involving the derivatives of functions. For our purposes, we will make two basic distinctions.

> **Definition: Pure-Time Differential Equation**
> A pure-time differential equation is a differential equation where the derivative of a function is given as an explicit function of the independent variable (ie. the function itself is only present as a derivative), generally assumed to be time.

An example of a pure-time differential equation is an equation of the form

$$\frac{dV}{dt} = \alpha$$

where $\alpha \in \mathbf{R}$. Without having developed more sophisticated methods, guessing a solution is our best means of solving an equation of this form. We need to 'guess' a function with a derivative that is a constant, α. Based on our knowledge of the derivative, we know that any linear function with a slope α will have a derivative of α, so any function of the form

$$V(t) = \alpha t + \beta$$

where $\beta \in \mathbf{R}$ will be a solution to this differential equation. When we differentiate $V(t)$ the constant β vanishes, so we can choose any value β and we will still have a solution. Since all of these solutions have the same functional form (lines with slope α), they are sometimes referred to as a family of solutions to this differential equation.

It is only possible to find a unique solution (a representative within the family of solutions) if we have an initial condition. An initial condition provides us with a numerical value for either the function or derivative at some point. For instance, if we had an initial condition that $V(0) = 2$, then we would need to find a function in our family that satisfies both $V(0) = 2$ and $\frac{dV}{dt} = \alpha$. Applying the initial condition to this family of functions we find

$$V(0) = \alpha \cdot 0 + \beta = 2$$

which implies that $\beta = 2$. Thus, $V(t) = \alpha t + 2$ is the function which satisfies both the differential equation *and* the initial condition. The problem of finding a specific solution to a differential equation based on initial conditions is called an initial value problem.

Example 1 Solve the differential equation

$$\frac{dP}{dt} = e^{-t}$$

with the initial condition $P(0) = 0$.
Solution The exponential e^t is a function with the same form as its derivative. In this case we need to choose a function with derivative e^{-t}, so we know it must some multiple of the exponential e^{-t}. Since

$$\frac{d}{dt}e^{-t} = -e^{-t}$$

we need to choose a function of the form $-e^{-t}$, so that the negative signs cancel. Once again, we can shift this function by any constant and still have a solution to the differential equation. We find that our solutions belong to the family

$$P(t) = -e^{-t} + \beta$$

where $\beta \in \mathbf{R}$. To find which specific function in the family our solution is, we need to use the initial condition $P(0) = 0$. We find

$$P(0) = -e^{-0} + \beta = -1 + \beta = 0$$

Thus, we find $\beta = 1$. Therefore, our exact solution is $P(t) = 1 - e^{-t}$.

We can also consider a second type of differential equation, called an autonomous differential equation.

Definition: Autonomous Differential Equation
An autonomous differential equation is a differential equation which does not explicitly contain the independent variable, which is often assumed to be time.

An example of an autonomous differential equation is the equation

$$\frac{db}{dt} = \lambda b$$

Note that in this equation there are no explicit references to t, and we are looking at a function b which has a rate of change proportional to the function itself. Recall that

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

and notice that $e^{\lambda t}$ is exactly the solution we need to this differential equation. In general, adding a constant to a solution to an autonomous differential equation will not yeild a new solution (the derivative of $e^{\lambda x} + \beta$ is not proportional to $e^{\lambda x} + \beta$, as the β vanishes upon differentiation). In this case, constant multiples of this solution are also solutions, so the family of solutions is

$$b(t) = \beta e^{\lambda t}$$

where $\beta \in \mathbf{R}$. Given some initial condition we could find a restriction on β to find an exact solution to the equation.

Euler's Method

Euler's method is a numerical method for solving initial value problems. Euler's method is based on the insight that a differential equation provides us with the slope of the function (at all points), and the initial value provides us with a point on the function. Using this information we can make a tangent line approximation to the function at that point. As we have seen, the tangent line is only a good approximation over a small interval. Thus, after moving a small interval, we will want to construct a new tangent line.

Although we know the value of the derivative of the function at the new point, we do not know the exact value of the function. Thus, we construct an approximate tangent line, using the slope of the function, and an approximation to the value of the function at the base point; we find the approximate value of the function using the value given by the previous tangent line. Continuing in this fashion, we construct a piecewise-linear approximation to the solution of the differential equation. We can also think of our approximation as a discrete function, which is defined for these approximated points. To make this discrete function a continuous function, we interpolate linearly between each pair of these points (draw a line between them).

Suppose we wish to solve the initial value problem

$$\frac{dm}{dt} = f(t)$$

with initial condition $m(t_0) = m_0$. The construction of a discrete function $\hat{m}(t)$ to approximate a solution is given as follows. Begin by setting $\hat{m}(t_0) = m(t_0) = m_0$. We will define $\hat{m}(t)$ on intervals of Δt, so it will next be defined at $\hat{m}(t_0 + \Delta t)$, then $\hat{m}(t_0 + 2\Delta t)$, and so on. We can find $\hat{m}(t + \Delta t)$ given $\hat{m}(t)$ using the following rule

$$\hat{m}(t + \Delta t) = \hat{m}(t) + m'(t)\Delta t$$

where $m'(t)$ is given by the differential equation in the initial value problem.

This is a discrete function which can be made continuous through linear interpolation. This continous function is defined on intervals, and for any point in the interval $[t_0, t_0 + \Delta t]$ the function is given by

$$\hat{m}(t_0 + t) = \hat{m}(t_0) + m'(t_0)t$$

where $t \in [0, \Delta t]$ (this restriction upon t ensures that $t_0 + t \in [t_0, t_0 + \Delta t]$). This is exactly the equation of the approximate tangent line to the function $m(t)$ at the point t_0 (called the approximate tangent line because we use $\hat{m}(t_0)$ rather than $m(t_0)$, which is an unknown, for our base point). By substituting Δt into this equation we can verify

$$\hat{m}(t_0 + \Delta t) = \hat{m}(t_0) + m'(t_0)\Delta t$$

which is consistent with our discretely defined $\hat{m}(t)$.

It is often easiest to solve problems of this type using a chart. We will outline this process in the following examples.

Example 1 Apply Euler's method to the differential equation

$$\frac{dV}{dt} = 2t$$

within initial condition $V(0) = 2$. Approximate the value of $V(1)$ using $\Delta t = 0.25$.
Solution We begin by setting $\hat{V}(0) = 2$. Next we construct the chart

t	$\hat{V}(t)$	$V'(t) = 2t$	$\hat{V}(t + \Delta t) = \hat{V}(t) + V'(t)\Delta t$
0	2	$2 \cdot 0 = 0$	$2 + 0 \cdot 0.25 = 2$
0.25	2	$2 \cdot 0.25 = 0.5$	$2 + 0.5 \cdot 0.25 = 2.125$
0.5	2.125	$2 \cdot 0.5 = 1$	$2.125 + 1 \cdot 0.25 = 2.375$
0.75	2.375	$2 \cdot 0.75 = 1.5$	$2.375 + 1.5 \cdot 0.25 = 2.75$
1	2.75	$2 \cdot 1 = 2$	$2.75 + 2 \cdot 0.25 = 3.25$

So we find that $V(1) \approx 3.25$. It turns out that $V(t) = t^2 + 2$ is the analytical solution to this differential equation, and $V(1) = 3$, so Euler's method provides a somewhat reasonable approximation, which could be greatly improved upon by decreasing the size of Δt.

Example 2 Apply Euler's method to the differential equation

$$\frac{dP}{dt} = e^{-t}$$

with the initial condition $P(0) = 0$. Approximate the value of $P(2)$.
Solution We begin by setting $\hat{P}(0) = 0$. We will use the time step $\Delta t = 0.5$. Next we construct the chart

t	$\hat{P}(t)$	$P'(t) = e^{-t}$	$\hat{P}(t + \Delta t) = \hat{P}(t) + P'(t)\Delta t$
0	0	$e^{-0} = 1$	$0 + 1 \cdot 0.5 = 0.5$
0.5	0.5	$e^{-0.5} \approx 0.60653$	$0.5 + 0.60653 \cdot 0.5 = 0.80326$
1	0.80326	$e^{-1} \approx 0.36788$	$0.80326 + 0.36788 \cdot 0.5 = 0.98721$
1.5	0.9872	$e^{-1.5} \approx 0.22313$	$0.9872 + .22313 \cdot 0.5 = 1.09877$
2	1.09877	$e^{-2} \approx 0.13534$	$1.09877 + 0.13534 \cdot 0.5 = 1.16644$

We have already solved for the analytical solution, $P(t) = 1 - e^{-t}$, so we find $P(2) = 0.86466$, which is somewhat lower than our approximation. By shrinking the size of the interval Δt, we could calculate a more accurate estimate.