Discrete-Time Dynamical Systems

Suppose we measure changes in a system over a period of time, and notice patterns in the data. If possible, we'd like to quantify these patterns of change into a dynamical rule - a rule that specifies how the system will change over a period of time. In doing so, we will be able to predict future states of the system. We quantify this change in terms of an updating function, which takes the current state of the system as an input, and specifies what the state of the system will be after some interval of time, an interval which corresponds to the specific updating function we are using. In general, an updating function is written

$$m_{t+1} = f(m_t)$$

Where m_t is the current state of the system, and m_{t+1} is the state of the system after one interval of time has passed. By iterating the above calculation multiple times, we can find subsequent states of of the system. For instance

$$f \circ f(m_t) = f(f(m_t)) = f(m_{t+1}) = m_{t+2}$$

Above we are composing the updating function with itself in order to update the state of the system twice, or find the state after two intervals of time have passed. With an understanding of the basic mechanics of how an updating function works, the next question to be answered is how to find an updating function. Suppose you measure the population of some species at the beginning of the year, and at the end of a year, with the following data

Initial Population b_0	Final Population, b_1
10	20

For such a set of data, we could describe the relationship between the two points in many ways: $b_1 = 2 \cdot b_0$, $b_1 = b_0 + 10$, etc. The bottom line is that in order to find the real relationship that describes the population change after a year has passed, we need to measure a few more years of data. Suppose we do so and find

Year	Initial Population	Final Population
1	10	20
2	20	40
3	40	80

In the above data we would say the initial population, before any years have passed, is b_0. We can say that between subsequent years the populations are related by $b_{t+1} = 2 \cdot b_t$. Having found a relationship between the initial and final population, we can construct an updating function

$$f(b_t) = 2 \cdot b_t$$

Example 1 Using the updating function $f(b_t) = 2 \cdot b_t$, as well as the above table of populations, calculate the population of the observed species after four and seven years of observation have passed (b_4 and b_7).

Solution We know the population after three years have passed ($b_3 = 80$). In order to calculate the population after four years have passed, we apply the updating function to b_3. Thus

$$b_4 = f(b_3) = 2 \cdot b_3 = 2 \cdot 80 = 160$$

We could apply the updating function many more times to find b_7, but it might be faster to observe the general trend between populations, and find a closed-form solution to the dynamical system. The closed form solution will allow us to calculate the population at the end of any year directly, rather than applying the updating function. Notice that $b_1 = 2 \cdot b_0$, so $b_2 = 2 \cdot b_1 = 2 \cdot 2 \cdot b_0 = 2^2 \cdot b_0$ and in general, $b_n = 2^n \cdot b_0$. Having found the closed form solution, we can simply calculate

$$b_7 = 2^7 \cdot b_0 = 2^7 \cdot 10 = 128 \cdot 10 = 1280$$

Example 2 Consider another colony of the species that is governed by the updating function $f(b_t) = 2 \cdot b_t$. For this colony, the initial population $b_0 = 5$. Find b_1, b_2, b_3, b_4, and b_7.
Solution There are two ways to approach this problem. We could begin as before by applying the updating function to find the subsequent popluations. For instance

$$b_1 = f(b_0) = 2 \cdot b_0 = 10$$

However, rather than iterating this updating function many times, we can use the closed-form solution that we found above to calculate these values directly.

$$b_2 = 2^2 \cdot b_0 = 4 \cdot 5 = 20$$
$$b_3 = 2^3 \cdot b_0 = 8 \cdot 5 = 40$$
$$b_4 = 2^4 \cdot b_0 = 16 \cdot 5 = 80$$
$$b_7 = 2^7 \cdot b_0 = 128 \cdot 5 = 640$$

If we plot the populations found in examples 1 and 2, we see that the population in example 1 is growing more quickly than in example 2, because the initial population (the initial condition) in that situation was greater. If the updating function is a multiplication, then the larger the initial condition is, the quicker the output will grow.

Example 3 Now suppose that you measure the change of the height of a few trees (in meters) over the course of a year, and come up with the following results

Tree	Initial Height	Final Height	Change in Height
1	10	11.1	1.1
2	20	21	1
3	15	15.9	0.9
4	25	26	1

Find a dynamical rule and corresponding updating function that describes the average change of the height of a tree during the course of a year.
Solution To find the average change in height of a tree during the course of a year, let us take the average of the change in heights of the four trees we measured.

$$\frac{1.1 + 1 + 0.9 + 1}{4} = 1$$

Given that the average change of tree height during a year is 1m, we find the dynamical rule

$$b_{h+1} = h_t + 1$$

This corresponds to the updating function

$$f(h_t) = h_t + 1$$

Example 4 Suppose two trees (T and P) grow each year according to the above dynamical rule. T has an initial height of 10m, whereas P has an initial height of 20m. Find the closed form solution for the height of the tree after t years have passed, and use it to plot the growth of T and P.
Solution Looking at the updating function, we can see that each year the tree grows by 1m. Thus, after t years have passed, the tree will have grown by t meters. So

$$h_t = h_0 + t$$

is the closed form solution to this dynamical system.

As we can see in plot of the height of the tree over a number of years, the difference in height between the trees T and P remains constant, and it is a distance of 10m - the distance between their initial heights.

Example 5 Suppose a patient is being treated for some disease. Suppose we know that the concentration of medication in the patient's bloodstream is halved every day. To counteract this, the patient is given a dose of medicine to raise the concentration in his or her bloodstream by 1 milligram per liter. Find an updating function to describe this dynamical system. Find the closed-form solution to this system.
Solution Let M_t describe the concentration after t days have passed. In order to describe the concentration after a day has passed, we will need to half the concentration from the previous day, and add 1 to it. Thus

$$f(M_t) = 0.5 \cdot M_t + 1$$

In order to find the closed-form solution, let us iterate this calculation multiple times, and look for a pattern.

$$M_1 = 0.5 \cdot M_0 + 1$$
$$M_2 = 0.5 \cdot M_1 + 1 = 0.5 \cdot (0.5 \cdot M_0 + 1) + 1 = 0.5^2 \cdot M_0 + 1 + 0.5$$
$$M_3 = 0.5 \cdot M_2 + 1 = 0.5 \cdot (0.5^2 \cdot M_0 + 1 + 0.5) + 1 = 0.5^3 \cdot M_0 + 1 + 0.5 + 0.5^2$$

As we can see, this is a rather complicated rule. However, we can describe the closed form solution using summation notation (for $t \geq 1$).

$$M_t = 0.5^t \cdot M_0 + \sum_{n=1}^{t} \frac{1}{2^{n-1}}$$

By plugging values into the above equation, we can calculate the concentration of medicine in the patient's bloodstream after any number of days have passed. If we evaluate the above expression as $t \to \infty$ (the patient has been treated with the drug for a very long time), we see that the expression 0.5^t becomes very small, so the initial concentration of the drug becomes irrelevant. Furthermore, using calculus to evalue the sum, we see that it approaches the value of 2. Thus, no matter what the value of M_0 is, eventually the concentration of the medicine will equilibrate to 2 milligrams per liter. As the above analysis was rather difficult, it might be preferable to start with an initial concentration and use the updating function multiple times, rather than searching for a closed-form solution. Doing so, one could see that the concentration is approaching 2.

Example 6 Suppose we have a population of bacteria that behaves according to the dynamical rule $b_{t+1} = 2 \cdot b_t$. Express this dynamical rule in terms of mass rather than population.
Solution The total mass of a colony of bacteria $m = \mu b$ where μ is the mass of a single bacterium and b is the population. Thus, the mass at time t can be expressed as

$$m_t = \mu \cdot b_t$$

and the updated mass can be expressed as

$$m_{t+1} = \mu \cdot b_{t+1} = \mu \cdot 2b_t = 2 \cdot m_t$$

Example 7 Suppose we have the following dynamical rule $h_{t+1} = h_t + 1$ for tree height in terms of meters. Find the corresponding rule in terms of cm.
Solution Let us define H_t to represent the height in terms of cm, so $H_t = 100 \cdot h_t$ (as 100 cm = 1m). It follows

$$H_{t+1} = 100 \cdot h_{t+1} = 100 \cdot (h_t + 1) = 100 \cdot h_t + 100 = H_t + 100$$