A Model of the Beating of a Heart

The beating of the human heart is regulated by the sinoatrial and atrioventricular nodes of the heart. The sinoatrial node acts as a pacemaker, or clock signal, for the heart, sending impulses at regular intervals of time in order to keep the heart beating. These signals pass through the atria to the sinoatrial node, which makes sure the heart is ready to beat before it tells the ventricles to contract and push blood out of the heart into the rest of the body. A single beat of the heart can be described as a step-by-step process as follows

1. The sinoatrial node fires an electrical pulse.
2. Current from the pulse causes the atria contract, pushing blood into the ventricles.
3. The signal reaches the atrioventricular node, and there is a slight delay, to allow the atria to empty.
4. If the heart is ready to beat, the atrioventricular node sends a signal to the ventricles.
5. The ventricles contract, pushing blood out of the heart to the rest of the body.
6. The heart muscle relaxes, the atria refill, and the atrioventricular node recharges.

The contraction of the atria and ventricles in response to electrical impulses is called systole, and the period of recovery after systole is called diastole. This process of the beating heart can be monitored using an electrocardiogram (abbreviated EKG or ECG). By attaching various electrodes to the human body, changes in the electric potential (voltage) across the heart can be measured, which correspond to different parts of the process of the beating of the heart. A normal EKG has a P wave, QRS complex, and T wave, which correspond to the contraction of the atria, the contraction of the ventricles, and the repolarization of the heart (the AV node recharging), respectively. The S-T segment between the QRS complex and the T wave corresponds to the slight delay after the signal reaches the AV node, which allows the atria to empty completely.

Our goal is to model a simplified version of the heart, represented in the following model

Here the sinoatrial node is acting as a clock, which sends signals to the atrioventricular node at regular intervals. The atrioventricular node regulates these signals, and if the heart is ready to beat, it sends a signal for the heart to beat (for the ventricles to contract).

The atrioventricular node uses as electric potential (voltage) in order to monitor the condition of the heart. Let us denote this potential as $V_t$, corresponding to the voltage after the $t^{th}$ signal from the sinoatrial node. Let $\tau$ be the interval of time between signals from the atrioventricular node,
and $\alpha$ be the decay rate of the atrioventricular node’s voltage. Based on this information, we can find the voltage of the AV node just before the next signal is received, $\tilde{V}_t$. We find that

$$\tilde{V}_t = e^{-\alpha \tau} V_t$$

To determine whether or not the heart is ready to beat, this voltage is compared to a threshold (critical) voltage $V_c$. There are two cases

- If $\tilde{V}_t = e^{-\alpha \tau} V_t \leq V_c$ or $V_t \leq e^{\alpha \tau} V_c$, then $V_{t+1} = \tilde{V}_t + u = e^{-\alpha \tau} V_t + u$ (i.e. the heart beats, and the voltage of the AV node is increased by $u$).
- If $\tilde{V}_t = e^{-\alpha \tau} V_t > V_c$ or $V_t > e^{\alpha \tau} V_c$, then $V_{t+1} = \tilde{V}_t$ (i.e. the heart does not beat).

We can encapsulate both of these results into a piecewise updating function

$$V_{t+1} = \begin{cases} e^{-\alpha \tau} V_t + u & V_t \leq e^{\alpha \tau} V_c \\ e^{-\alpha \tau} V_t & V_t > e^{\alpha \tau} V_c \end{cases}$$

In summary, the atrioventricular node regulates heartbeats with a decaying voltage. Once the voltage has decayed enough ($\tilde{V}_t \leq V_c$) the heart is ready to beat. When it beats, the voltage of the AV node is increased by $u$, in order to ‘reset’ this decaying voltage. For convenience, let us define

$$c = e^{-\alpha \tau}$$

Since the argument of the exponential function is negative, the only possible values of $c$ are $0 \leq c \leq 1$. If $c \approx 0$ then there is a very fast decay ($\alpha$ large), and if $c \approx 1$ there is a very slow decay ($\alpha$ small).

In terms of the new parameter $c$, the updating function is

$$V_{t+1} = \begin{cases} cV_t + u & V_t \leq \frac{1}{c} V_c \\ cV_t & V_t > \frac{1}{c} V_c \end{cases}$$

Now let us cobweb the updating function for the parameter values of $u = 1$, $V_c = 1$ Volt, and $c = 0.4$, $c = 0.6$. 
Since we are cobwebbing a piecewise updating function, there is a ‘break’ in the graph of the function. This corresponds to the voltage $e^{\alpha \tau} V_c$, which is the threshold condition for beating and not beating of the heart. The first branch of the updating function corresponds to beating, whereas the second corresponds to skipping a beat.

The heart normally beats in a periodic pattern, which is its steady-state condition (a condition that is not changing with time). However, these cobweb diagrams begin elsewhere and then move into a steady-state condition. Thus, these cobweb diagrams were generated by a perturbation on the heart, which removed it from its steady state condition. As we cobweb this updating function, we are watching the heart subsequently return to its steady-state behavior (normal beating).

In the case where there is an equilibrium on the first branch, the heart with beat normally, and stay in equilibrium indefinitely. If there were an equilibrium on the second branch (the ‘skip’ branch) it would correspond to the condition where the heart was never beating, or dead. In the other case cobwebbed above ($c = 0.6$), the AV node recovers too slowly between signals, which leads to a condition known as second-degree block.

We are in equilibrium when $V_{t+1} = V_t$. This will only occur after we complete a full cycle, as we need the AV voltage to be increased by $u$ in order to counteract its decay. Thus

$$V^* = cV^* + u$$

$$V^* = \frac{u}{1 - c}$$

Note that this equilibrium can only occur when the heart is ready to beat at each signal, i.e.
In the case where there is no equilibrium, we have a second degree block. In a 2:1 AV block the heart beats with every other signal, so we go through two cycles for every beat of the heart. At the first signal, the heart is not ready to beat, and at the second signal the voltage of the AV node has decayed sufficiently for the heart to beat. This situation is illustrated above in the cobweb diagram with \( c = 0.6 \). In this case, based on the parameters of \( u \) and \( V_c \), \( c \) is too large (so the AV voltage decays too slowly) for regular beating of the heart.

Another situation in which there is irregular beating is called the Wenckebach phenomenon. Left untreated, this condition often leads to a 2:1 AV block. In the Wenckebach phenomenon, the recovery rate of the heart is just above equilibrium, so the heart is barely able to recover after each beat. After a number of beats, the heart is then unable to recover quickly enough, so it skips a beat. Then it continues this pattern (although this behavior is periodic, it is important to remember that it does not correspond to an equilibrium). On an EKG, this corresponds to a lengthening PQ interval, in which it eventually becomes long enough that it forces the heart to skip a beat. This is represented as follows on a cobwebbing diagram.

In actual measurements of this phenomenon there is a slight slowing of the beats just before a beat is skipped. This discrepancy occurs because our model does not account for the rise time of the sinoatrial signal. If the AV node is a low voltage, it will be able to respond near the beginning of this build up, but if it is a potential right near threshold, the SA signal will need to build up more before the AV node can respond, whereby the atrioventricular node responds more slowly near threshold.