l’Hôpital’s Rule

In simple cases we are able to evaluate limits of the form

\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} \]

where \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \infty \) or \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \) using knowledge of the different rates at which functions approach \( \infty \) or 0, respectively. However, if we are faced with a more complicated limit, such as

\[ \lim_{x \to 0} \frac{e^{3x} - 1}{x} \]

it is not clear at what relative rates the numerator and denominator approach 0. When we are faced with a limit of

\[
\begin{array}{c|c}
0 & \infty \\
\hline
0 & \infty \\
\end{array}
\]

the limit is said to be in an indeterminate form. In general, to evaluate limits of this type we will need to use l’Hôpital’s Rule.

**l’Hôpital’s Rule**

Suppose that \( f(x) \) and \( g(x) \) are differentiable on an open interval \( I \) containing \( x_0 \), and that

\[ \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm \infty \]

or

\[ \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \]

Further, suppose that \( g'(x) \neq 0 \) for all \( x \neq x_0 \) in \( I \). It follows that

\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \]

provided that the right limit exists, or is \( \pm \infty \).

Essentially, l’Hôpital’s rule tells us that if a limit is in an indeterminate form, we can evaluate the limit by looking at the derivatives of the functions involved in the limit, provided that the derivatives exist. If we are faced with another indeterminate form, we can once again apply l’Hôpital’s rule, provided that the second derivatives exist. In most cases, we should eventually find a limit that is not in an indeterminate form, which will tell us the limit of the original two functions. It should be emphasized that if a limit is not in an indeterminate form, then l’Hôpital’s rule cannot be applied.

**Example 1** Evaluate the limit

\[ \lim_{x \to 0} \frac{e^{3x} - 1}{x} \]
**Solution** As stated before, we will need to use l'Hôpital’s rule to evaluate this limit. By inspection, we can see that both the numerator and denominator are differentiable, so it is okay to apply the rule. Thus, we find
\[
\lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{x \to 0} \frac{3e^{3x}}{1} = 3
\]

**Example 2** Evaluate the limit
\[
\lim_{x \to \infty} \frac{e^{3x} - 1}{x}
\]
**Solution** In this situation, we know that \(e^x\) approaches \(\infty\) faster than \(x\) as \(x \to \infty\). Thus, the limit should be \(\infty\). Let us verify this using l'Hôpital’s rule.
\[
\lim_{x \to \infty} \frac{e^{3x} - 1}{x} = \lim_{x \to \infty} \frac{3e^{3x}}{1} = \infty
\]

**Example 3** Evaluate the limit
\[
\lim_{x \to 0} \frac{e^{3x}}{x}
\]
**Solution** In this case we do not have an indeterminate form, so we cannot apply l'Hôpital’s rule. As \(x \to 0\), the numerator approaches 1, while the denominator approaches 0, so we find that
\[
\lim_{x \to 0} \frac{e^{3x}}{x} = \infty
\]

**Example 4** Evaluate the limit
\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1}
\]
**Solution** First notice that this limit is in the indeterminate form \(0/0\), so we can apply l'Hôpital’s rule.
\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2
\]
Alternatively, we can write \(x^2 - 1 = (x + 1)(x - 1)\), and cancel a factor of \(x - 1\) from the numerator and denominator. Then, we find
\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 2
\]

**Example 5** Evaluate the limit
\[
\lim_{x \to 0} \frac{3x - \sin(x)}{x}
\]
**Solution** This limit is in the indeterminate form \(0/0\), so we can apply l'Hôpital’s rule.
\[
\lim_{x \to 0} \frac{3x - \sin(x)}{x} = \lim_{x \to 0} \frac{3 - \cos(x)}{1} = 2
\]

**Example 6** Evaluate the limit
\[
\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2}
\]
Solution Once again, this limit is in the indeterminate form $0/0$, so we can apply l'Hôpital's rule.

\[
\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2} = \lim_{x \to 0} \frac{(1/2)(1 + x)^{-1/2} - 1/2}{2x} = \lim_{x \to 0} -\frac{(1/4)(1 + x)^{-3/2}}{2} = -\frac{1}{8}
\]

Example 7 Evaluate the limit

\[
\lim_{x \to \infty} \frac{\ln(x)}{2\sqrt{x}}
\]

Solution Now we have an indeterminate form $\frac{\infty}{\infty}$. We find

\[
\lim_{x \to \infty} \frac{\ln(x)}{2\sqrt{x}} = \lim_{x \to \infty} \frac{1}{x} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0
\]