A Model of Gas Exchange in the Lungs

Suppose we have a lung that has a volume of 3.0 L when full, and 2.4 L after exhalation. With each exhalation 0.6 L of air is expelled from the lung, and each inhalation brings 0.6 L of ambient air (from the outside). Through this process, the air in the lungs mixes with air that is brought in through inhalation.

Suppose we wish to track the amount of some chemical in the lungs throughout this breathing process. Let c_t describe the concentration of this chemical in the lungs. Let the ambient air have a concentration of 5.0 mmol/L (a mole is $6.02 \cdot 10^{23}$ molecules, so a millimole is $6.02 \cdot 10^{20}$ molecules). To find the total amount of this molecule, we can use the relation

$$\text{total amount} = \text{concentration} \cdot \text{volume}$$

Similarly, we can use the total amount to find the concentration

$$\text{concentration} = \frac{\text{total amount}}{\text{volume}}$$

We will make two simplifying assumptions before we proceed.

1. The air in the lung is completely mixed with each breath. This means that concentration of the molecule in exhaled air is equal to the concentration of molecule in the lung as a whole.
2. No chemical is produced or consumed during this process. Thus, the lung is acting as a mechanism which mixes air, but nothing more.

While both of these assumptions are untrue, they will make it simpler to create our first model of the lung (we can be more accurate in a second or third model). We will follow through the processes of this system using a table.

Step	Volume (L)	Concentration (mmol/L)	Total Chemical (mmol)
Air in lung before breath	3.0	c_t	$3.0c_t$
Air exhaled	0.6	c_t	$0.6c_t$
Air in lung after exhalation	2.4	c_t	$2.4c_t$
Air inhaled	0.6	5.0	3.0
Air in lung after breath	3.0	$1.0 + 0.8c_t$	$3.0 + 2.4c_t$

Throughout this table we look at the volume of the air, and multiply it by the concentration in the environment to find the total amount of the chemical. To find the total amount in the lung after a breath, we add the total amount of chemical in the lung after exhalation and the amount added with inhalation. From there we can divide by the volume (3.0) to find the concentration. Thus, we have found a discrete-time dynamical system described by

$$c_{t+1} = 1.0 + 0.8c_t$$

Let us solve for the equilibria of this sytem (c^*)

$$
\begin{aligned}
c^* &= 1.0 + 0.8c^* \\
c^* - 0.8c^* &= 1.0 \\
0.2c^* &= 1.0 \\
c^* &= 5.0
\end{aligned}
$$

It is left as an exercise to the reader to verify the same through cobwebbing.

In the previous model made assumptions about the volume of the lung, and the amount of air exhaled and inhaled during a breath. By parametrizing these values, we can generalize the model to numerous situations. Let us suppose the lung has a volume of V L, W L of air are inhaled and exhaled, and that the ambient chemical concentration is γ. Now we can follow the same analysis as before.

Step	Volume (L)	Concentration (mmol/L)	Total Chemical (mmol)
Air in lung before breath	V	c_t	$V \cdot c_t$
Air exhaled	W	c_t	$W \cdot c_t$
Air in lung after exhalation	$V - W$	c_t	$(V - W) \cdot c_t$
Air inhaled	W	γ	$W \cdot \gamma$
Air in lung after breath	V	$\frac{(V-W) \cdot c_t + \gamma W}{V}$	$(V - W) \cdot c_t + \gamma W$

The analysis done above is completely analogous to that done with the unparametrized system. This yields the result

$$c_{t+1} = \frac{(V - W) \cdot c_t + \gamma W}{V}$$

This result can be simplified by separating the terms on the right-hand side

$$c_{t+1} = (1 - \frac{W}{V})c_t + \gamma \frac{W}{V}$$

Now we find that W and V only appear as the ratio $\frac{W}{V}$, the fraction of the total volume exchanged with each breath. Let us define $q = \frac{W}{V}$. Then

$$c_{t+1} = (1 - q)c_t + q\gamma$$

Let us solve for the equilibria for this system, as we did with the original system.

$$
\begin{aligned}
c^* &= (1 - q)c^* + q\gamma \\
0 &= (1 - q)c^* - c^* + q\gamma \\
0 &= q(\gamma - c^*) \\
q = 0 \quad &\text{or} \quad c^* = \gamma
\end{aligned}
$$

Thus, the equilibrium point is $c^* = \gamma$. That is, the system is in equilibrium when the concentration of the molecule in the lung is equal to the ambient concentration. If $q = 0$, then no air is exchanged with each breath, so the concentration does not change. If $q = 1$, then all of the air in the lung is exhaled, and the concentration in the lung becomes the ambient concentration with inhalation (which will bring the system into equilibrium after one breath).

In the above system, the concentration after a breath is a weighted average of the previous concentration, and the ambient concentration. It is weighted with the percentage of the total volume exchanged q. This brings us to the following definition.

Definition: Weighted Average
A weighted average between two values x and y is a sum of the form $qx + (1-q)y$, where $0 \leq q \leq 1$. If $q = \frac{1}{2}$ then this is an ordinary average.

Example 1 Let $p = 1$ and $r = 6$. Suppose a weighted average puts a weight of q on p and a weight of $(1-q)$ on r. Find the average for $q = 0.25, 0.5, 0.75$.
Solution For $q = 0.25$ we have

$$qp + (1-q)r = 0.25 \cdot 1 + 0.75 \cdot 6 = 4.75$$

which is closer to r than p. For $q = 0.5$ we have

$$qp + (1-q)r = 0.5 \cdot 1 + 0.5 \cdot 6 = 3.5$$

which is an ordinary average of p and r. For $q = 0.75$ we have

$$qp + (1-q)r = 0.75 \cdot 1 + 0.25 \cdot 6 = 2.25$$

which is closer to p than r.

Example 2 Suppose 2 L of liquid with a concentration of 20.0 mmol/L of salt are mixed with 4 L of liquid with a concentration of 2 mmol/L of salt. What is the resulting concentration?
Solution The result solution is 6L of liquid, which contains 2 L of the liquid with a high concentration of salt ($\frac{1}{3}$ of the total volume) and 4 L of the liquid with a low concentration of salt ($\frac{2}{3}$ of the total volume). Thus the concentration is described by

$$\frac{1}{3} \cdot 20 \, \frac{\text{mmol}}{\text{L}} + \frac{2}{3} \cdot 2 \, \frac{\text{mmol}}{\text{L}} = \frac{24}{3} \, \frac{\text{mmol}}{\text{L}} = 8 \, \frac{\text{mmol}}{\text{L}}$$

Example 3 Suppose we are mixing the solutions as in the above example, but add a 2 L solution of 4 mmol/L concentration. Now find the resulting concentration.
Solution We can proceed just as before, but must add a third term, and realize that each solution is now a smaller portion of the total volume. This highest concentration solution is now $\frac{1}{4}$ the total volume, the new concentration is also $\frac{1}{4}$ and the lower concentration is $\frac{1}{2}$ the total concentration. Thus

$$\frac{1}{4} \cdot 20 \, \frac{\text{mmol}}{\text{L}} + \frac{1}{2} \cdot 2 \, \frac{\text{mmol}}{\text{L}} + \frac{1}{4} \cdot 4 \, \frac{\text{mmol}}{\text{L}} = 6 \, \frac{\text{mmol}}{\text{L}}$$

Thus far, our models have not accounted for a very important part of respiration - absorption. Now let us suppose the molecule we are interested in within the lung and the ambient air is oxygen. Let's use the same model as before, with the addition that before exhalation, a fraction α of the oxygen in the lung is absorbed. This leads us to the following system

$$c_{t+1} = (1-q)(1-\alpha)c_t + q\gamma$$

Note that if $\alpha = 0$, ie. there is no absorption, then this is the same system as before. Once again, we can find the equilibrium concentration

$$\begin{aligned}
c^* &= (1-q)(1-\alpha)c^* + q\gamma \\
c^* - (1-q)(1-\alpha)c^* &= q\gamma \\
\big(1-(1-q)(1-\alpha)\big)c^* &= q\gamma \\
c^* &= \frac{q\gamma}{1-(1-q)(1-\alpha)}
\end{aligned}$$

Example 4 Suppose $q = 0.2$, $\alpha = 0.3$, and $\gamma = 0.21$ mmol/L. Find the equilibrium concentration.
Solution Using the above result for the equilibrium concentration

$$c^* = \frac{q\gamma}{1-(1-q)(1-\alpha)} = \frac{0.042}{1-0.8\cdot 0.7} = \frac{0.042}{0.44} \approx 0.095$$

Recall that if there was no absorption, then we would find $c^* = 0.21$. Thus, accounting for absorption, we find that the equilibrium concentration reduced by more than 50%.

Example 5 Suppose $q = 0.2$, $\gamma = 0.21$ mmol/L, and we find an equilibrium concentration of $c^* = 0.15$ mmol/L. Find α.
Solution Use the equation for the equilibrium concentration and solve for α.

$$\begin{aligned}
0.15 &= \frac{0.042}{1-0.8(1-\alpha)} \\
0.15(1-0.8(1-\alpha)) &= 0.042 \\
0.2 + 0.8\alpha &= \frac{0.042}{0.15} \\
\alpha &= \frac{0.28-0.2}{0.8} \\
\alpha &= 0.1
\end{aligned}$$