The Second Derivative

When we take the derivative of a function $f(x)$, we get a derived function $f'(x)$, called the derivative or first derivative. If we now take the derivative of this function $f'(x)$, we get another derived function $f''(x)$, which is called the second derivative of f. In differential notation this is written $\frac{d^2f}{dx^2}$. If we think of $\frac{d}{dx}$ as an operator, we can think of $\frac{d^2}{dx^2}$ as representing the operator being applied twice. The second derivative of $f(x)$ tells us the rate of change of the derivative $f'(x)$ of $f(x)$.

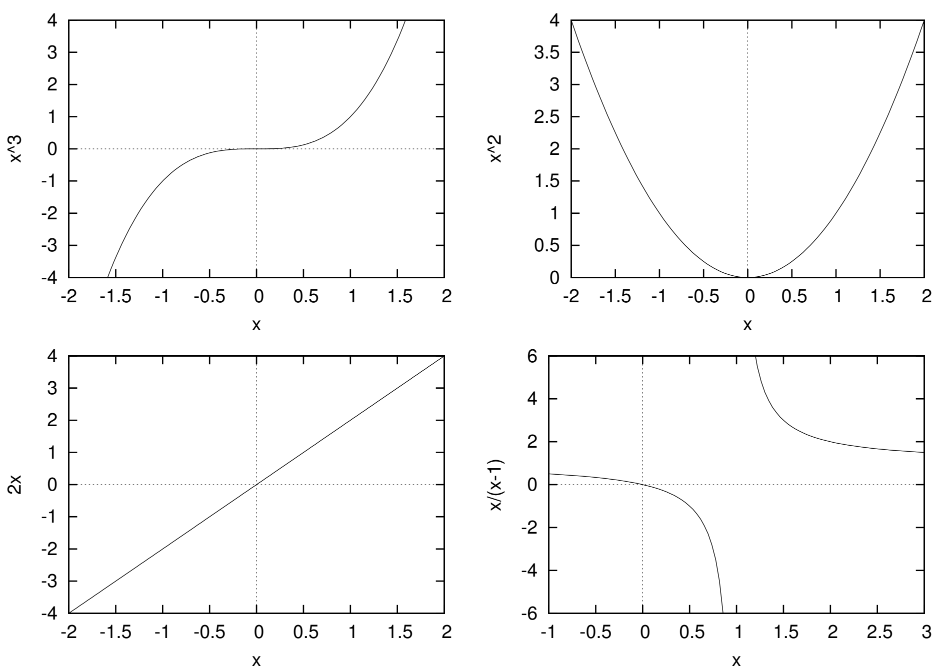
More specifically, the second derivative describes the curvature of the function f. If the function curves upward, it is said to be concave up. If the function curves downward, then it is said to be concave down. The behavior of the function corresponding to the second derivative can be summarized as follows

1. *The second derivative is positive* ($f''(x) > 0$): When the second derivative is positive, the function $f(x)$ is concave up.

2. *The second derivative is negative* ($f''(x) < 0$): When the second derivative is negative, the function $f(x)$ is concave down.

3. *The second derivative is zero* ($f''(x) = 0$): When the second derivative is zero, it corresponds to a possible inflection point. If the second derivative changes sign around the zero (from positive to negative, or negative to positive), then the point is an inflection point. This corresponds to a point where the function $f(x)$ changes concavity. If the second derivative does not change sign (ie. it goes from positive to zero to positive), then it is not an inflection point ($x = 0$ with $f(x) = x^4$ is an example of this).

Let us consider the following functions, and look at how their derivatives correspond to their graphs.



Example 1 Find $f'(x)$ and $f''(x)$ if $f(x) = x^3$. Compare these derivatives to the graph above.

Solution By repeated applications of the power rule, we find that $f'(x) = 3x^2$, and $f''(x) = 6x$. For all x, the first derivative $f'(x) > 0$, so the function $f(x)$ is always increasing. Considering the second derivative, we see that for $x < 0$ we have $f''(x) < 0$, so $f(x)$ is concave down. For $x > 0$ we have $f''(x) > 0$, so $f(x)$ is concave up. At $x = 0$, $f''(x) = 0$, and since the second derivative changes signs around 0, this is an inflection point, as can be seen above.

Example 2 Find $f'(x)$ and $f''(x)$ if $f(x) = x^2$. Compare these derivatives to the graph above.
Solution By repeated applications of the power rule, we find that $f'(x) = 2x$, and $f''(x) = 2$. For all x, the second derivative $f''(x) > 0$, so the function $f(x)$ is always concave up. Considering the first derivative, we see that for $x < 0$ we have $f'(x) < 0$, so $f(x)$ is decreasing. For $x > 0$ we have $f'(x) > 0$, so $f(x)$ is increasing. At $x = 0$, $f'(x) = 0$, which corresponds to a critical point, where $f(x)$ is not changing, and is in fact, the minimum of $f(x)$.

Example 3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x$. Compare these derivatives to the graph above.
Solution By repeated applications of the power rule, we find that $f'(x) = 2$, and $f''(x) = 0$. For all x, the first derivative $f'(x) > 0$, so the function $f(x)$ is always increasing. Also, for all x, the second derivative is 0. This corresponds to a graph that does not have any concavity, such as the line above.

Example 4 Find $f'(x)$ and $f''(x)$ if $f(x) = \frac{x}{x-1}$. Compare these derivatives to the graph above.
Solution In this situation we cannot just use the product rule to calculate the derivatives. Instead, we must use the quotient rule. Let $u(x) = x$ and $v(x) = x - 1$. We find that $u'(x) = 1 = v'(x)$, as these are both linear functions. We find the derivative as

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2} = \frac{x - 1 - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

In order to calculate the second derivative, we once again need to use the quotient rule. Let $u(x) = -1$ and $v(x) = (x-1)^2 = x^2 - 2x + 1$. We find that $u'(x) = 0$ and $v'(x) = 2x - 2$. Using the quotient rule we find

$$f''(x) = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2} = \frac{(x-1)^2 \cdot 0 - (-1)(2x-2)}{((x-1)^2)^2} = \frac{2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

In summary, $f'(x) = \frac{-1}{(x-1)^2}$ and $f''(x) = \frac{2}{(x-1)^3}$. Now that we've completed the arduous task of calculating the above derivatives, we can continue to compare them to the graph above. The derivative is negative for all $x \neq 1$, and is not defined for $x = 1$. This indicates that for all $x \neq 1$ the function $f(x)$ is decreasing. However, when we look at the graph of f, we see that values of f for $x > 1$ are greater than values of f for $x < 1$. This strange behavior occurs because the derivative is not defined for $x = 1$, where the function value essentially increases by an infinite amount. The behavior of $f''(x)$ is not so complicated, and it shows us that for $x < 1$ the function is concave down, as $f''(x) < 0$. For $x > 1$, $f''(x) > 0$, so the function is concave up. At $x = 0$ the function and its derivatives are not defined, so it makes little sense to talk about the influence of the second derivative. Nevertheless, this point is something of an inflection point (although not technically), as the concavity of the function changes here. This example illustrates that functions and their derivatives may have very unexpected behavior at points where their denominator's go to zero.

In the previous examples we found the derivatives and compared their behavior to the graphs of the function that we already knew. Based on this insight, we should be able to sketch a function based on knowledge of its derivatives. The general procedure is as follows

1. Find $f'(x)$. The points where $f'(x) = 0$ or is not defined are critical points. These are points where it is possible for the sign of $f'(x)$ to change. If the sign changes from positive to negative, then the point is called a local maximum. If the sign changes from negative to positive, the point is called a local minimum. By looking at the sign of the derivative between these points, we can map out the regions where the function is increasing and decreasing.

2. Find $f''(x)$. The points where $f''(x) = 0$ are possible inflections points. By looking at the sign of the second derivative around these points, we can map out the regions where the function is concave up and down, as well as determine which of these points are inflection points.

3. Check $f(x)$ for divisions by 0. These points correspond to vertical asymptotes. As the function approaches one of these vertical asymptotes, it will either increase or decrease without bound (approach $\pm\infty$).

4. Having examined the derivatives to find the minima, maxima, and inflection points, plot the function at these points. Now add any vertical asymptotes to the graph (if they exist). Finally, use the information about the regions between these points to sketch the function.

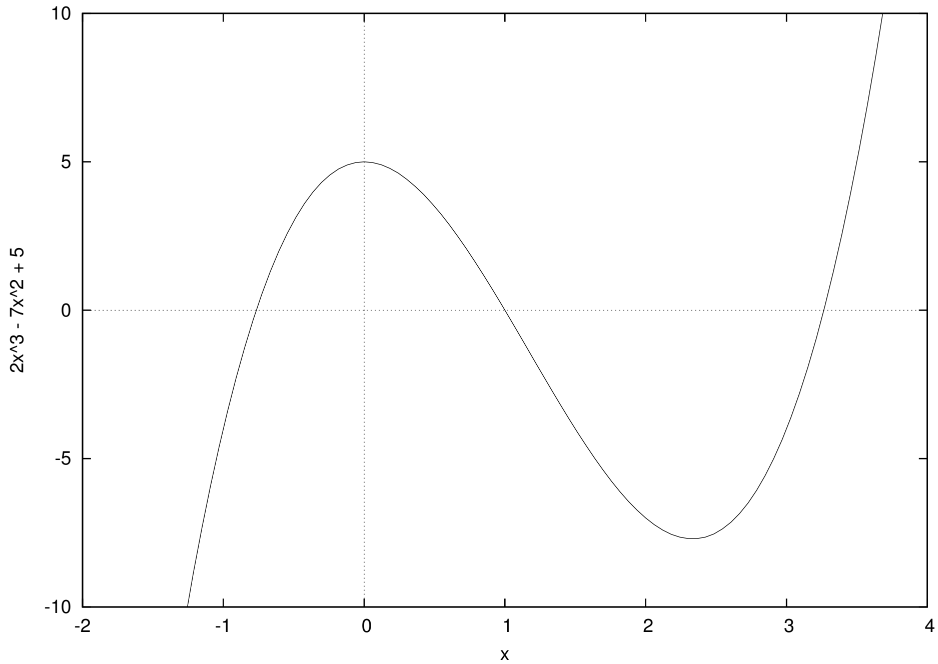
Example 5 Sketch $f(x) = 2x^3 - 7x^2 + 5$
Solution The first step is to find the first and second derivatives. In this case, we simply apply the product rule to find that $f'(x) = 6x^2 - 14x$ and $f''(x) = 12x - 14$. The first derivative is defined for all x, so in order to find the critical points let us solve for $f'(x) = 0$

$$0 = 6x^2 - 14x = x(6x - 14)$$

Thus we have critical points when $x = 0$ and $x = \frac{14}{6} = \frac{7}{3} = 2.\overline{3}$. These points divide $f'(x)$ into 3 intervals; within each we need to check the sign of the derivative (one point per interval is sufficient, as the derivative only changes sign at a critical point). By looking at $f'(x)$, we can see that for very large or small values of x, the positive quadratic term will dominate the negative linear term, making the first derivative positive, so the function is increasing in the first and last interval. Now we can use a sample point of $x = 1$, which lies between the two critical points, to find that $f'(1) = 6 - 18 = -8 < 0$. Thus, in the middle interval the function is decreasing.

In order to search for possible inflection points, we set the second derivative equal to 0, and solve for x. We find that $0 = 12x - 14$, so $x = \frac{14}{12} = \frac{7}{6} = 1.1\overline{6}$. Now we must compare the sign of the second derivative for $x > 1.1\overline{6}$ and $x < 1.1\overline{6}$ to see if it an inflection point or not. We find that for $x < 1.1\overline{6}$ we have $f''(x) < 0$, so the function is concave down. For $x > 1.1\overline{6}$ we have $f''(x) > 0$, so the function is concave up. Thus, it is truly an inflection point, and we know the concavity of the function changes from down to up at this point.

The final step is to plot the function at the above points of interest: $(0, f(0))$, $(1.1\overline{6}, f(1.1\overline{6}))$, and $(2.\overline{3}, f(2.\overline{3}))$, which correspond to $(0, 5)$, $(1.16, -1.35)$, and $(2.3, -7.7)$. Once we have plotted these key points, we simply need to connect the dots, and be cognizant of the concavity and whether or not the function is increasing or decreasing. The final graph is pictured below.



Our final application of the second derivative is less mathematical, and more physical. If we have a function of position, say $y(t)$, the first derivative corresponds to velocity, and the second derivative corresponds to acceleration. Thus, we can rewrite Newton's force equation as

$$F = ma = m\frac{d^2y}{dt^2}$$

If we know the force and the mass of the object, then the above is a differential equation which we can solve in order to find the acceleration, velocity, and finally position. Recall that a falling object has position described by

$$y(t) = 9.8t^2$$

velocity described by

$$v(t) = y'(t) = 19.6t$$

and finally acceleration described by

$$a(t) = y''(t) = 19.6$$

where the above quantities are measured in meters and seconds (velocity is meters per second, where acceleration is meters per second squared).