

Derivatives of Trigonometric Functions

The trigonometric functions are a final category of functions that are very useful in many applications. Rather than derive the derivatives for $\cos(x)$ and $\sin(x)$, we will take them axiomatically, and use them to find the derivatives of other trigonometric functions.

$$\frac{d}{dx} \sin(x) = \cos(x)$$

and

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

To remember which derivative contains the negative sign, recall the graphs of the sine and cosine functions. At $x = 0$, $\sin(x)$ is increasing, and $\cos(x)$ is positive, so it makes sense that the derivative is a positive $\cos(x)$. On the other hand, just after $x = 0$, $\cos(x)$ is decreasing, and $\sin(x)$ is positive, so the derivative must be a negative $\sin(x)$.

Example 1 Find all derivatives of $\sin(x)$.

Solution Since we know $\cos(x)$ is the derivative of $\sin(x)$, if we can complete the above task, then we will also have all derivatives of $\cos(x)$.

$$\frac{d}{dx} \sin(x) = \cos(x)$$

gives us the first derivative of the sine function.

$$\frac{d^2}{dx^2} \sin(x) = \frac{d}{dx} \cos(x) = -\sin(x)$$

gives us the second derivative. Also

$$\frac{d^3}{dx^3} \sin(x) = -\frac{d}{dx} \sin(x) = -\cos(x)$$

Finally

$$\frac{d^4}{dx^4} \sin(x) = -\frac{d}{dx} \cos(x) = \sin(x)$$

Now we can see that the fourth derivative of $\sin(x)$ is $\sin(x)$, so we can easily enough find any derivative of the sine function as follows. Suppose we want to find the n^{th} derivative of sine. All we need to do is divide n by 4, and look at the remainder r . If we take the r^{th} derivative of sine, it will be exactly the same as taking the n^{th} derivative, as every four derivatives will simply return us to the original result of the sine function. Applying this principle, we find that the 17^{th} derivative of the sine function is equal to the 1^{st} derivative, so

$$\frac{d^{17}}{dx^{17}} \sin(x) = \frac{d}{dx} \sin(x) = \cos(x)$$

The derivatives of $\cos(x)$ have the same behavior, repeating every cycle of 4. The n^{th} derivative of cosine is the $(n + 1)^{th}$ derivative of sine, as cosine is the first derivative of sine. The derivatives of sine and cosine display this cyclic behavior due to their relationship to the complex exponential function. Euler's formula states that $e^{ix} = \cos(x) + i \sin(x)$, so we find that $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$. Taking the derivative of these two equations provides an alternative method to

find the derivatives of sine and cosine, as well as elucidating the link between these elusive functions.

Knowledge of the derivatives of sine and cosine allows us to find the derivatives of all other trigonometric functions using the quotient rule. Recall the following identities:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \cot(x) = \frac{\cos(x)}{\sin(x)} \quad \sec(x) = \frac{1}{\cos(x)} \quad \csc(x) = \frac{1}{\sin(x)}$$

Example 2 Find the derivative of $\csc(x)$.

Solution First recall that $\csc(x) = \frac{1}{\sin(x)}$. Now let us apply the quotient rule

$$\frac{d}{dx} \csc(x) = \frac{\sin(x) \cdot 0 - 1 \cdot \cos(x)}{\sin(x)^2} = -\frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)} = -\cot(x) \csc(x)$$

Example 3 Find the derivative of $e^x \cdot \cos(x) + x^2$.

Solution To find this derivative, we will utilize the sum and product rules.

$$\frac{d}{dx}(e^x \cdot \cos(x) + x^2) = e^x \cdot \cos(x) + e^x \cdot (-\sin(x)) + 2x = e^x(\cos(x) - \sin(x)) + 2x$$

Example 4 Find the derivative of a general sinusoidal function.

Solution Recall that the form of a general sinusoidal function is

$$f(t) = A + B \cdot \cos\left(\frac{2\pi}{T}(t - \phi)\right)$$

Thus, in order to find the derivative we will need to apply the chain rule.

$$\frac{d}{dt}(A + B \cdot \cos\left(\frac{2\pi}{T}(t - \phi)\right)) = B\left(-\sin\left(\frac{2\pi}{T}(t - \phi)\right)\right) \cdot \frac{2\pi}{T} = -\frac{2B\pi}{T} \sin\left(\frac{2\pi}{T}(t - \phi)\right)$$

We can see in the above equation that the smaller the period T becomes, the larger the derivative is. This makes sense as the frequency is inverse of the period $f = \frac{1}{T}$, so the derivative above confirms our intuition that waveforms with high frequency change rapidly.

Example 5 The blood flow in a whale's arteries is pulsatile. Suppose the blood flow along the artery of a whale is

$$F(t) = 212 \cos\left(\frac{2\pi t}{10}\right)$$

where F is measured in liters per second and t is measured in seconds.

- a. Find the average flow and amplitude.
- b. Find the period of this flow. How many heartbeats does this whale have per minute?
- c. When is the flow zero? What is happening at these times?
- d. Find the rate of change of the flow. What does it mean when this is zero? What is the flow at these times?

Solution

- a. The average is 0 and the amplitude is 212 l/sec.

- b. The period is 10 seconds. This corresponds to a heartbeat every 10 seconds, which means the heart beats 6 times per minute.
- c. The flow is zero when $\frac{2\pi t}{10} = \frac{n\pi}{2}$, where n is an odd integer. Thus, it is zero when $t = \frac{5n}{2} = 2.5n$. At these times the blood has stopped flowing, and is changing direction of flow to the opposite direction.
- d. The rate of change of the flow is $F'(t) = -212 \frac{2\pi}{10} \sin(\frac{2\pi t}{10}) = -41.4\pi \sin(\frac{2\pi t}{10})$. The derivative is at a zero when $\frac{2\pi t}{10} = n\pi$, where n is an integer. Thus, it is zero when $t = 5n$. When $F'(t) = 0$, it means that the flow is at a maximum in one direction or the other. When $t = 5n$, n even, $F(t) = 212$ l/sec. When $t = 5n$, n odd, $F(t) = -212$ l/sec.

Example 6 At Barrow, Alaska, atmospheric CO_2 levels in parts per million can be modeled by

$$C(x) = 7.5 \sin(2\pi x) + .04x^2 + .6x + 330$$

where x is in years and $x = 0$ is the year 1960. Find $C'(x)$ for the year 2003. What does this mean?

Solution First let us find $C'(x)$.

$$C'(x) = 15\pi \cos(2\pi x) + 0.08x + 0.6$$

For the year 2003, we have $x = 43$. It follows that

$$C'(43) = 15\pi \cos(86\pi) + 0.08 \cdot 43 + 0.6 = 15 + 34.4 + 0.6 = 50$$

This result tells us that in the year 2003, the atmospheric CO_2 levels are increasing by 50ppm. This is compared to a rate of 15.6 ppm in the year 1960.