1. a. First note that the limit is of the indeterminate form $0/0$, so we can apply l'Hôpital's Rule. We find that
$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\lim_{x\to 0}\frac{\sin(x)}{2x}.$$
Once again we need to apply l'Hôpital's Rule. We find
$$\lim_{x\to 0}\frac{\sin(x)}{2x}=\lim_{x\to 0}\frac{\cos(x)}{2}=\frac{1}{2}.$$

   b. This limit is not in an indeterminate form, so we cannot apply l'Hôpital's Rule. Since x is increasing to 7, the denominator is negative, so we find that
$$\lim_{x\nearrow 7}\frac{4}{x-7}=-\infty.$$

   c. Since the order of the polynomial in the denominator is larger than that in the numerator, the limit is 0. Using l'Hôpital's Rule (noting this is the indeterminate form ∞/∞) we find
$$\lim_{x\to\infty}\frac{2x^2+3x}{x^3+x+1}=\lim_{x\to\infty}\frac{4x+3}{3x^2+1}=\lim_{x\to\infty}\frac{4}{6x}=0.$$

   d. Since we have an indeterminate form of $0/0$ we can apply l'Hôpital's Rule. We find
$$\lim_{x\searrow 0}\frac{2x}{x+7\sqrt{x}}=\lim_{x\searrow 0}\frac{2}{1+7/(2\sqrt{x})}.$$
To proceed we multiply by 1, finding that
$$\lim_{x\searrow 0}\frac{2}{1+7/(2\sqrt{x})}=\lim_{x\searrow 0}\frac{2}{1+7/(2\sqrt{x})}\cdot\frac{\sqrt{x}}{\sqrt{x}}=\lim_{x\searrow 0}\frac{2\sqrt{x}}{\sqrt{x}+7/2}=0.$$

   e. Noting that
$$\cot(x)=\frac{\cos(x)}{\sin(x)}\quad\text{and}\quad\csc(x)=\frac{1}{\sin(x)}$$
we find that
$$\frac{\cot(x)}{\csc(x)}=\frac{\cos(x)/\sin(x)}{1/\sin(x)}=\cos(x).$$
It follows that
$$\lim_{x\searrow 0}\frac{\cot(x)}{\csc(x)}=\lim_{x\searrow 0}\cos(x)=1.$$

2. To have a continuous function we must have
$$\lim_{x\to 0}f(x)=f(0)=c.$$
Considering this limit (which is in the indeterminate form $0/0$) we find
$$\lim_{x\to 0}\frac{9x-3\sin(3x)}{5x^3}=\lim_{x\to 0}\frac{9-9\cos(3x)}{15x^2}.$$
This is still in the indeterminate form $0/0$, so we apply l'Hôpital's Rule again. We find
$$\lim_{x\to 0}\frac{9-9\cos(3x)}{15x^2}=\lim_{x\to 0}\frac{27\sin(3x)}{30x}=\lim_{x\to 0}\frac{81\cos(3x)}{30}=\frac{81}{30}=2.7.$$
Thus, we choose $c=2.7$ in order to make the function continuous.

3. a. Noting that $f'(x) = g'(x) = 1$ we see that

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{1}{1} = 1.$$

Also, evaluating the limit

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x+2}{x+1} = \frac{2}{1} = 2.$$

b. This does not contradict l'Hôpital's Rule because the functions f and g in the limit as $x \to 0$ do not meet the hypotheses of l'Hôpital's Rule (they are not in an indeterminate form).