1. Newton's method fails because we have a horizontal tangent line which does not intersect the x-axis. Thus, there is no next approximation to the root.

2. Our goal is to find when $f(x) = x^4 - 2x^3 - x^2 + 2 = 0$. We have

$$f'(x) = 4x^3 - 6x - 2x.$$

It follows that for an approximation x_n we have

$$x_{n+1} = x_n - \frac{x_n^4 - 2x_n^3 - x_n^2 + 2}{4x_n^3 - 6x_n - 2x_n}.$$

If we start with $x_0 = 0.5$ we find $x_4 = 0.630115396$, and starting from $x_0 = 2.5$ we find $x_4 = 2.57327196$.

3. Here we want to find solutions to $f(x) = \tan(x) - 2x = 0$. We find

$$f'(x) = \sec^2(x) - 2.$$

It follows that for an approximation x_n we have

$$x_{n+1} = x_n - \frac{\tan(x) - 2x}{\sec^2(x) - 2}.$$

We should choose a starting approximation between $[0, \pi/2]$, say $x_0 = 1$. This yields $x_1 = 1.31048$, and eventually

$$x \approx 1.16556.$$

4. Recall that $3^{1/7}$ is defined as the number x such that

$$x^7 = 3.$$

Thus, we want to find a solution to the equation

$$f(x) = x^7 - 3 = 0.$$

We find that

$$f'(x) = 7x^6,$$

so we have

$$x_{n+1} = x_n - \frac{x^7 - 3}{7x^6}.$$

Noting that $1^7 = 1$ and $2^7 = 128$, we can make an educated starting guess near $x = 1$, say $x_0 = 1.2$. Doing so we find that $x_1 = 1.1721$, and eventually

$$3^{1/7} = x \approx 1.1699308.$$

We can be confident that our approximation is accurate up to 4 decimal places of accuracy by looking at how the approximations change over time as we move from x_n to x_{n+1}. After the first 4 decimal places stop changing for awhile we can be confident that our approximation is accurate at least up to 4 decimal places.

5. a. Since the submarine moves along the parabola $y = x^2$, the distance between the buoy and the submarine along its trajectory at some point (x, y) is given by

$$D(x) = \sqrt{(x-2)^2 + (y-(-1/2))^2} = \sqrt{(x-2)^2 + (y+1/2))^2}.$$

Noting that $y = x^2$ we have

$$D(x) = \sqrt{(x-2)^2 + (x^2+1/2)^2}.$$

Noting that both quantities under the radical are positive, minimizing this quantity is the same as minimizing

$$f(x) = (x-2)^2 + (x^2+1/2)^2.$$

To minimize this distance we look at the first derivative,

$$f'(x) = 2(x-2) + 2(x^2+1/2) \cdot 2x = 2x - 4 + 4x^3 + 2x = 4(x^3 + x - 1).$$

This has a critical point when

$$x^3 + x - 1 = 0,$$

or

$$x = \frac{1}{x^2+1}.$$

Noting that the second derivative

$$f''(x) = 4(3x^2+1) > 0,$$

we see that this is a minimum.

b. We want to solve for

$$f'(x) = 4(x^3 + x - 1) = 0.$$

Thus, we have

$$x_{n+1} = x_n - \frac{4(x_n^3 + x_n - 1)}{4(3x_n^2+1)}.$$

Starting with an initial approximation of $x_0 = 1$ we find

$$x \approx 0.682327.$$