0.1 Concavity and Curve Sketching

The standard technique for sketching a function is plotting points and connecting the dots. However, there are issues to consider such as which points to plot and how to connect them. Consider a sine function. If all of the points we choose happen to be at the peaks of the function, say $\pi/2$, $5\pi/2$, etc., then upon connecting our dots we will just get a straight line, which is not like a sine function at all. Similarly, if all of the points we plot are at alternating peaks, $\pi/2$, $3\pi/2$, $5\pi/2$, etc., when we connect the dots we will get a jagged sawtooth curve - also inaccurate. Although this plot is more accurate than the original one, it is still missing a very important feature - curvature. Curvature is also called concavity.

Over a given interval, a function can either be concave up, concave down, or have no concavity. A function is concave up when it is curved upwards, like a bowl, concave down when it is curved downward like a hill, and is neither if it has no curvature. Referencing the sine function, it is concave up over the interval $[\pi, 2\pi]$, where is it shaped like a bowl, and concave down over the interval $[0, \pi]$, where is it shaped like a hill. There are no intervals over which the sine function has no concavity or curvature. For an example of a function without concavity, consider a straight line.

Concavity, or curvature of a function is related to a changing steepness of the curve. Thus, in order to determine the concavity of a function we need to know something about the magnitude of its first derivative, but we can’t simply evaluate the first derivative at all points within an interval, as there are too many points to do so. We’ll have to use a different method. Consider the function $x^3$ (see figure 1).

![Figure 1: The graph of a function that is both concave up and down.](image)

First note that this function is concave down for $x < 0$ and concave up for $x > 0$. Also notice that $f'(x) = 3x^2$, so the function $f(x) = x^3$ is increasing at all points other than $x = 0$ (because $3x^2 > 0$ for all $x \neq 0$). As $x \to 0$, we see that the function begins to increase more slowly, corresponding to a decreasing first derivative, and as $x$ becomes larger than 0, the rate at which the function increases increases, corresponding to an increasing first derivative. The changing magnitude of the first derivative lends curvature to the function.

**Definition 0.1.1 (Concavity).** We say a function $f$ is *concave up* on an interval $I$ if $f'$ is increasing on $I$. Similarly, we say a function $f$ is *concave down* on an interval $I$ if $f'$ is decreasing on $I$.

From the above definition we have translated a visual property of how a function is curved into a mathematical property, related to a changing first derivative. Rather than trying to evaluate the first derivative of a function at many points over an interval, we look at the second derivative, which tells us the rate of change of the first derivative.
Theorem 0.1.1 (Second Derivative test for Concavity). Let $f$ be a twice-differentiable function.

1. If $f'' > 0$ on $I$, then $f$ is concave up on $I$.
2. If $f'' < 0$ on $I$, then $f$ is concave down on $I$.

In addition to identifying the intervals over which a function is concave up and down, we are interested in identifying the points where concavity can possibly change. A point where the concavity of a function changes is called an inflection point. Using similar arguments as those made with critical points, we can see that if either $f''(c) = 0$ or $f''$ does not exist, then $c$ is a possible inflection point. If the second derivative changes sign around one of these points (from positive to negative, or negative to positive), then the point is an inflection point, which is a point where the function $f(x)$ changes concavity. If the second derivative does not change sign (i.e., it goes from positive to zero to positive), then it is not an inflection point ($x = 0$ with $f(x) = x^4$ is an example of this).

Before we move onto using concavity as a part of curve sketching, we note that using a function’s concavity can be a helpful tool for classifying its extrema. Consider a function $f$ with $f'(c) = 0$. Now suppose $f'' > 0$ on an interval around the critical point $c$, which implies the first derivative is continuous and increasing on this interval. It follows that the first derivative must begin negative in order to increase to 0, and become positive after increasing beyond 0. In other words, such a point is a local minimum. There is a similar result for a negative second derivative, which we summarize in the following theorem.

Theorem 0.1.2 (Second Derivative Test for Local Extrema). Suppose $f'(c) = 0$, and $f''$ is continuous on an open interval around $x = c$.

1. If $f''(c) < 0$, then $f$ has a local maximum at $x = c$
2. If $f''(c) > 0$, then $f$ has a local minimum at $x = c$
3. If $f''(c) = 0$, then the test is inconclusive. $f$ may have a local maximum, minimum, or neither at $x = c$.

Why do we require that $f''$ be continuous in the hypothesis of the above theorem? The reason is as follows. If $f''$ is continuous, then the fact that $f''(c) > 0$ or $f''(c) < 0$ implies that $f'' > 0$ or $f'' < 0$ on some interval around $x = c$, because the function is continuous so it can’t jump. It’s noteworthy that in many cases we cannot apply this theorem. First, we cannot use it for critical points at which the first derivative does not exist. Second, it does not give us any information for when the second derivative is 0 as well (such as $x^3$, $x^4$, etc). Finally, it is not useful for classifying endpoints. Despite these restrictions it is sometimes useful, for the simple fact that when we want to graph a function, we need to find the second derivative anyway.

Now equipped with our knowledge of concavity we can move on to sketching functions. The basic procedure will to be to start with the first derivative to identify the critical points of the function. This will partition the domain of our function into subintervals over which it is increasing, decreasing, or neither. Next we will want to identify the inflection points, dividing the domain further into subintervals with different concavities. Finally, we evaluate the function at the critical points, inflection points, and end points, connecting the dots using our knowledge of the way the function changes.

Example 1. Sketch $f(x) = x^3$ on $[-2, 2]$ using the first and second derivatives.
Solution We are already familiar with the graph of this function, so this will be a bit of a warmup. By repeated applications of the power rule, we find that

\[ f'(x) = 3x^2 \quad \text{and} \quad f''(x) = 6x. \]

Since \( f'(0) = 0 \) and \( f''(0) = 0 \), we have that \( x = 0 \) is a critical point and a possible inflection point, but we cannot use the second derivative test because \( f''(0) = 0 \). Evaluating the function at the critical points and endpoints, we find that \( f(-2) = -8 \), \( f(0) = 0 \), and \( f(2) = 8 \). Since the first derivative cannot change signs on the subinterval \((-2, 0)\), and \( f(0) > f(-2) \), it follows that the first derivative is positive on \((-2, 0)\). Likewise, the first derivative is positive on \((0, 2)\), so \( f(0) \) is not an extreme value. Considering the second derivative, we see that for \( x < 0 \) we have \( f''(x) < 0 \), so \( f(x) \) is concave down. For \( x > 0 \) we have \( f''(x) > 0 \), so \( f(x) \) is concave up. Since the second derivative exists and changes signs around 0, \( x = 0 \) is an inflection point. Now we can plot the endpoints and the critical point, and use our knowledge of the second derivative to connect the dots.

Example 2. Sketch \( f(x) = x^2 \) on \([−3, 3]\) using the first and second derivatives.

Solution By repeated applications of the power rule, we find that

\[ f'(x) = 2x \quad \text{and} \quad f''(x) = 2. \]

Thus, we have a critical point at \( x = 0 \), and evaluating the function at the critical points and endpoints we find \( f(-3) = 9 \), \( f(0) = 0 \), and \( f(3) = 9 \). It follows from the mean value theorem that the function is decreasing on \((-3, 0)\) and increasing on \((0, 3)\), which is confirmed by looking at the first derivative. It follows \( f(0) \) is a local minimum. Also, using the second derivative test we see \( f''(0) > 0 \), which implies \( f(0) \) is a local minimum. In fact, for all \( x \), the second derivative \( f''(x) > 0 \), so the function \( f(x) \) is always concave up. Now we plot the endpoints and critical points, connecting the dots with the knowledge that the function is always concave up.

Example 3. Find \( f'(x) \) and \( f''(x) \) if \( f(x) = 2x \). Use this to analyze the familiar graph of the function.

Solution We find

\[ f'(x) = 2 \quad \text{and} \quad f''(x) = 0. \]

For all \( x \), the first derivative \( f'(x) > 0 \), so the function \( f(x) \) is always increasing. Also, for all \( x \), the second derivative is 0. This corresponds to a graph that does not have any concavity, which is the familiar line.

Example 4. Sketch \( f(x) = \frac{x}{x-1} \) on \([−5, 5]\) using the first and second derivatives.

Solution The first thing to notice is that we have division by 0 at \( x = 1 \), so it is possible that our function could increase or decrease without bound as \( x \to 1 \). We find that

\[ \lim_{x \to 1} \frac{x}{x-1} = -\infty \quad \text{and} \quad \lim_{x \to 1} \frac{x}{x-1} = \infty. \]

In order to aid ourselves in sketching the graph of this function, we can draw a vertical dotted line at \( x = 1 \), called a vertical asymptote. This will provide us a guideline for sketching the fact that our function is increasing and decreasing without bound around this point of interest. Next, we need to calculate the first and second derivatives of \( f \). Using the quotient rule we find

\[ f'(x) = \frac{(x-1) \cdot 1 - x \cdot 1}{(x-1)^2} = \frac{-1}{(x-1)^2} = -(x-1)^{-2}. \]
Now we can use the power rule and chain rule, to find
\[ f''(x) = 2 \cdot (x - 1)^{-3} \cdot 1 = \frac{2}{(x - 1)^3}. \]

Now we can use the above derivatives to analyze the behavior of the function. Since both the first and second derivative are fractions in which the numerator is constant, neither can equal zero. Thus, our only point of interest is \( x = 1 \), where the first and second derivative are not defined, because the function itself is not defined at \( x = 0 \). Since the denominator is nonnegative, the first derivative is negative for all \( x \neq 1 \). This indicates that for all \( x \neq 1 \) the function \( f(x) \) is decreasing.

Looking at the second derivative we see that for \( x < 1 \) the function is concave down, as \( f''(x) < 0 \). For \( x > 1 \), \( f''(x) > 0 \), so the function is concave up. Even though the sign of the second derivative changes around \( x = 1 \), since the second derivative is not defined there, we do not call \( x = 1 \) an inflection point. Evaluating the function at the end points, we see \( f(-5) = 5/6 \) and \( f(5) = -5/4 \). Even with this much information, it is difficult to construct an accurate graph of this function. The simplest way to construct a more accurate sketch is to plot some additional points, such as \((0, f(0) = 0)\) and \((2, f(2) = 1)\) (see figure 2).

![Figure 2: Graph of \( f(x) = \frac{x}{x-1} \).](image)

**Example 5.** Sketch \( f(x) = 2x^3 - 7x^2 + 5 \)

**Solution** The first step is to find the first and second derivatives. We find that
\[ f'(x) = 6x^2 - 14x \quad \text{and} \quad f''(x) = 12x - 14. \]
The first derivative is defined for all \( x \), so in order to find the critical points we solve \( f'(x) = 0 \), finding
\[ 0 = 6x^2 - 14x = x(6x - 14). \]
Thus, we have critical points when \( x = 0 \) and \( x = 14/6 = 7/3 = 2.3 \). These points divide \( f'(x) \) into 3 intervals:
\[ (-\infty, 0), \quad (0, 7/3), \quad \text{and} \quad (7/3, \infty). \]

Now we need to check the sign of the derivative over each of these subintervals (one point per interval is sufficient, as the derivative only changes sign at a critical point). By looking at \( f'(x) \), we can see that for very large or small values of \( x \), the positive quadratic term will dominate the negative first-order term, making the first derivative positive, so the function is increasing in the
first and last interval. Now we can use a sample point of $x = 1$, which lies between the two critical points, to find that $f'(1) = 6 - 14 = -8 < 0$. Thus, in the middle interval the function is decreasing.

In order to search for possible inflection points, we set the second derivative equal to 0, and solve for $x$. We find that $0 = 12x - 14$, so $x = 14/12 = 7/6 = 1.16$. Now we must compare the sign of the second derivative for $x > 1.16$ and $x < 1.16$ to see if it an inflection point or not. We find that for $x < 1.16$ we have $f''(x) < 0$, so the function is concave down. For $x > 1.16$ we have $f''(x) > 0$, so the function is concave up. Thus, it is truly an inflection point, and we know the concavity of the function changes from down to up at this point.

The final step is to plot the function at the above points of interest: $(0, f(0))$, $(1.16, f(1.16))$, and $(2.3, f(2.3))$, which correspond to $(0, 5)$, $(1.16, -1.35)$, and $(2.3, -7.7)$. Once we have plotted these key points, we simply need to connect the dots, and be cognizant of the concavity and whether or not the function is increasing or decreasing. The final graph is pictured in figure 3.

![Graph of $2x^3 - 7x^2 + 5$](image.png)

**Figure 3:** $2x^3 - 7x^2 + 5$. 