0.1 Infinite Limits

The concept of infinity (or symbolically, $\infty$) plays an important role in calculus. This concept is related to the boundedness of a function.

**Definition 0.1.1 (Bounded Function).** A function $f : X \to \mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$ so that for all $x \in X$,

$$|f(x)| \leq M.$$

We say $f$ is bounded above if there exists $M \in \mathbb{R}$ so that for all $x \in X$,

$$f(x) \leq M.$$

We say $f$ is bounded below if there exists $M \in \mathbb{R}$ so that for all $x \in X$,

$$f(x) \geq M.$$

In each of these cases $M$ is said to be a bound, upper bound, or lower bound, respectively. If no such $M$ exists, then a function is said to be unbounded, unbounded above, or unbounded below, respectively.

The basic idea behind boundedness is creating a wall (either a ceiling or floor) that the function cannot pass. If a function is bounded above then we can create a ceiling above the function, so that the value of the function is always below the ceiling. For a function that is bounded below, we can construct a floor below it, so that the value of the function is always above the floor. For a bounded function, we can construct both a ceiling above it and a floor below it. The smallest number that works as an upper bound is called the least upper bound or supremum, and the largest number that works as a lower bound is called the greatest lower bound or infemum. Other than remarking that these concepts are extremely important in analysis, we will not focus on them here.

The notion of infinity is contrary to that of boundedness; infinity represents unboundedness. The exact definition of infinity is contextual, but in general $\infty$ represents something increasing without bound (which means there is no upper bound for the function), and $-\infty$ represents something decreasing without bound (which means there is no lower bound for the function). Before proceeding further, we need to emphasize that $\pm \infty$ are not numbers. Since $\infty$ and $-\infty$ are not numbers, performing arithmetic operations with them is nonsensical.

**Claim 0.1.1 ($\infty$ as a number is nonsense).** Suppose there exists a real number $\infty$, which is larger than all other real numbers. It follows that $0 = \infty$, which is nonsense.

**Proof.** Let us begin by multiplying $\infty$ by 2. Since there is no number larger than $\infty$, it is clear that the result must be $\infty$. Thus,

$$2 \cdot \infty = \infty.$$

It follows that

$$2 \cdot \infty - \infty = \infty - \infty = 0.$$

because for any number $x$, $x - x = 0$. We can also write that

$$2 \cdot \infty - \infty = (2 - 1)\infty = \infty.$$

Finally, we conclude that

$$0 = \infty,$$

by equating the above expressions. Thus, the number which is largest in magnitude is equal to the number which is smallest in magnitude, which is nonsense.
We were able to draw such a nonsensical conclusion in the above illustration because we manipulated $\infty$ as a number, which it is not. Since $\pm \infty$ are not numbers, we will never encounter functions that have a value of $\pm \infty$ at some point. The question then remains, when do we encounter $\pm \infty$? In order to proceed further, we need to be more specific. There are two primary contexts in which the notion of infinity will be useful, and both are related to limits.

Consider the function

$$f(x) = \frac{1}{x}.$$ 

If we look as $x \searrow 0$, the value of the function becomes larger and larger (increasing without bound), because the fraction is positive, and the magnitude of its denominator becomes smaller and smaller. On the other side, as $x \nearrow 0$, the values of the function become smaller and smaller (decreasing without bound), because the denominator is negative, and its magnitude is approaching 0. It is not possible for this function to have a (finite) limit from either side, as on the right it becomes larger than every finite number, and on the left it becomes smaller than any finite number. Not only does this function not have one-sided limits as $x \to 0$, the limits do not exist for very specific reasons - either because the function is increasing or decreasing without bound. In order to represent this using the concept of infinity, we write

$$\lim_{x \nearrow 0} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \searrow 0} \frac{1}{x} = \infty.$$ 

In general, when we write $\lim_{x \to x_0} f(x) = \infty$, we are not saying that the limit exists; we are saying is that the limit does not exist, and it does not exist because the values of the function $f(x)$ grow without bound as $x \to x_0$. Similarly,

$$\lim_{x \to x_0} f(x) = -\infty$$

means that the limit as $x \to x_0$ of $f(x)$ does not exist because the function values decrease without bound (become arbitrarily large in magnitude, and negative in sign). Throughout this discussion one-sided limits will be particularly useful to us, because it is quite common that a function may approach $\infty$ from one side of a point, and $-\infty$ from the other side of the point, just as the function $f(x) = 1/x$ does.

**Definition 0.1.2 (Infinite Limits).** We write

$$\lim_{x \to x_0} f(x) = \infty$$

if for every number $B > 0$ there exists a corresponding $\delta > 0$ such that for all $x$ with $0 < |x - x_0| < \delta$ we have

$$f(x) > B.$$ 

Similarly, we write

$$\lim_{x \to x_0} f(x) = -\infty$$

if for every number $B > 0$ there exists a corresponding $\delta > 0$ such that for all $x$ with $0 < |x - x_0| < \delta$ we have

$$f(x) < -B.$$
It is important to emphasize that the rules we previously had for combining limits do not apply to limits as \( f(x) \to \pm\infty \), because these limits do not actually exist. If we try to apply the rules for combining finite limits, we will run into many nonsensical possibilities, such as \( \infty - \infty \), \( \infty / \infty \), etc., which are meaningless, because \( \infty \) is not a symbol to be manipulated as a number.

Limits where \( f(x) \to \pm\infty \) have an important physical interpretation in terms of frequency response. For this reason, we will write \( f \) as \( f(\omega) \), using \( \omega \) to denote we have a function of frequency. The frequency response of a system gives the output of the system with respect to the frequency of the input signal. In general, a system responds differently to inputs coming in at different frequencies. The types of systems we are interested in are oscillating systems, those in which some value is moving repetitively between a minimum and maximum. Oscillating systems are important in the fields of physics, mechanics, and electromagnetics. Examples are the simple harmonic oscillator (mass on a spring or pendulum), electrical circuits, electromagnetic waves, etc. Places where the (ideal or undamped) frequency response of a system approaches \( \pm\infty \) (or in a real physical situation, becomes very large) are called resonant frequencies, and the phenomenon of the output of the system becoming very large in reaction to an input at a certain frequency is called resonance.

A simple physical example related to frequency response is pushing a child on a swing. If one pushes the swing every time the child moves all the way back, the swing will start moving rapidly, elevating the child higher and higher. If we look at the separation between each push, we can think of the pushes occurring with a certain frequency. Now what happens if we change that frequency, by pushing more frequently, or less frequently? If we start pushing the child in the middle of the motion of the swing, our input forces will act destructively rather than constructively, and not be very effective at swinging the child (just think about what would happen if you pushed the child in the middle of a swing, rather than at the end of it). Here the resonant frequency is the one at which we need to push the child in order for the swinging to occur effectively.

In the above example there are other factors to consider, such as dampening in the chains of the swing, and the physical setup of the swing. These impose practical limitations on the speed and height at which the swing could be pushed. For this reason the response of the system at the resonant frequency is not an infinite output, merely an output much larger than if the input force were at a different frequency. Only in an ideal case (such as a model which ignores certain physical phenomena) will the frequency response of the system ever be \( \pm\infty \). However, if in a model we have an infinite frequency response, we know that in practice we will have a large response, unless there are significant dampening factors to control the resonance.

In electrical circuits resonance can be used for tuning signals. A circuit is setup so that its frequency response is much higher at a given frequency than others, so effectively only that frequency (and the narrow band of frequencies around it) is transmitted (a simple RLC circuit tunes signals in this way). Another example of resonance is in a laser cavity, which consists of two mirrors and a gain medium in between them. As light oscillates between the two mirrors, some frequencies undergo constructive interference, whereas others undergo destructive interference, which frequencies of light depending on the length of the optical cavity. The result is that certain frequencies are amplified greatly - the resonant frequencies - and others are not amplified. As a result, a beam of light with a very narrow frequency spectrum can be generated, which is called a laser. If we look at the frequency response of the laser cavity, we can see which frequencies resonate, and thus determine what frequency lasers can be generated with it. Of course there are other issues to consider, such as which frequencies are amplified by the gain medium, how to get enough energy into the cavity, and how much power can be sustained by the cavity before the mirrors are destroyed.

Just as resonance can be exploited for positive effects, it is also a hazard to avoid. In mechanical systems there is a worry about the system vibrating at the resonant frequency, until the forces...
become so great that the machine is destroyed. For this reason it is important to include dampening
to curb the infinite (ideal) responses of the system, until they are controlled to the point where they
are not self-destructive. A famed example of resonance as a hazard is the Tacoma Narrows bridge
failure of 1940. Although this is not an example of forced resonance, it is a related phenomenon.

In addition to considering the behavior of a physical system as its output grows considerably
large, it is also interesting to look at the behavior of a system after a consider amount of time
has passed. The definition of considerable depends on the nature of the system, because it is not
physically possible (or necessarily useful) for us to observe a system for an arbitrarily large amount
of time. Unless a system is chaotic, over time it should fall into some sort of steady state behavior.
This is what leads us to the second usage of the concept of infinity.

We can look at the behavior of a function (representing a physical system) after a considerable
amount of time has passed by looking at what happens as the input (time) grows without bound;
i.e. we consider the limit as \( t \to \infty \). If the independent variable is not time, we might look at limits
as \( x \to \pm \infty \). Even though a concept of infinite time is meaningless physically, a limit as \( t \to \infty \)
describes what would happen in a system, were it possible to let it continue on indefinitely. We call
the behavior of a function as its input approaches \( \pm \infty \) the asymptotic behavior of the function.

**Definition 0.1.3** (Definition: Limits as \( x \to \infty \) or \( x \to -\infty \)). We write

\[
\lim_{x \to \infty} f(x) = L
\]

if for every number \( \varepsilon > 0 \) there exists a corresponding number \( M \) such that for all \( x \) with \( x > M \)
we have

\[
|f(x) - L| < \varepsilon.
\]

Similarly, we write

\[
\lim_{x \to -\infty} f(x) = L
\]

if for every number \( \varepsilon > 0 \) there exists a corresponding number \( N \) such that for all \( x \) with \( x < N \)
we have

\[
|f(x) - L| < \varepsilon.
\]

The method for showing a limit exists as \( x \to \pm \infty \) is very similar to showing that a limit exists
at a point. The only difference is that rather than creating a \( \delta \) interval around the point of interest
to restrict the function values within the error tolerance of the limit value, we need to find a value
\( M \) or \( N \) so that if the input values \( x \) are larger/smaller than \( M \) or \( N \), the function values are
within the error tolerance of the limit. Due to the remarkable similarities between these notions
of the limit, it may not be surprising that all of the properties we stated previously for combining
limits also hold for limits at \( \pm \infty \). We will not focus on proofs of limits as \( x \to \pm \infty \).

A function may exhibit many possible behaviors in the limits as its argument \( x \to \pm \infty \). The
function can either approach \( \pm \infty \), meaning that it increases or decreases without bound, it can
approach a constant value, or it might not approach any value. For instance,

\[
\lim_{x \to \infty} x = \infty,
\]

because of the output of a function is its input, so as the input grows without bound, so does the
output. If we have a constant function, \( f(x) = c \), then the input approaching \( \infty \) is irrelevant, so
we find

\[
\lim_{x \to \infty} c = c.
\]
Finally, we may encounter functions that oscillate with time, and never settle down to a single value, or increase/decrease without bound. Here we do not use specific notation to represent that the limit does not exist, we simply write

\[ \lim_{x \to \infty} \cos(x) \text{ does not exist.} \]

Nevertheless, if we modulate the cosine function with a decreasing amplitude, then we will have a function that approaches 0.

\[ \lim_{x \to \infty} x^{-2}\cos(x) = 0. \]

For any of the above limits, if we multiply the function by \(-1\), the sign will simply change; if a function increases without bound, the negative of that function will decrease without bound. Thus,

\[ \lim_{x \to \infty} -x = -\infty. \]

Similarly, if we look in the limit as \( x \to -\infty \), the only thing we might have to worry about is a negative sign - otherwise the analysis is the same.

In the next section we will consider how to analyze functions and determine whether or not they approach \( \pm \infty \) in certain limits.