0.1 Lengths of Plane Curves

In addition to using integration to calculate area and volume, we can use integration to measure length. For the time being we will focus only on two-dimensional curves. For a general curve in a two-dimensional plane it is not clear exactly how to measure its length. In everyday physical situations one can place a string on top of the curve, and then measure the length of the string when it is straightened out, noting that the length of the string is the same whether it is wound up or not. Unfortunately, we have no means of running a string over an arbitrary curve, one that we might not even be able to sketch. Instead, we need to use the notion of approximation, and use a limit to make the approximation as accurate as we would like. The simplest means of approximating a curve is using straight line segments. As we increase the number of segments, they begin to hang closer and closer to the curve, and in the limit that the number of segments approaches infinity, we find the exact length of the curve.

The first issue to resolve is how to represent a general curve in a two-dimensional plane. Although we can use a function $f(x)$ to represent a curve, the number of curves we can represent using just functions is rather limited. Instead, we need to shift our focus to parameterized curves. Essentially, a parameterized curve consists of two time-dependent functions, $x(t)$ and $y(t)$, where one represents the $x$-coordinate of the curve at a given time, and the other the $y$-coordinate. The collection of points $(x(t), y(t))$ defines the curve itself. Since we are defining the curve in terms of the independent variable of time, we can actually think of the curve as representing the trajectory of some point, and as time increases the point moves along the curve.

In order to proceed in finding the lengths of two-dimensional curves we will need to impose some slight restrictions on the curves we are interested in dealing with. First of all, there is no guarantee that two arbitrary functions $x(t)$ and $y(t)$ will in any way define a nice curve. They may define some set of points filled with jumps, breaks, etc, that do very little to represent an actual curve. The first restriction we need to make is to consider plane curves.

**Definition 0.1.1 (Plane Curve).** Let $x$ and $y$ be continuous functions on $[a, b]$. The set of points

$$\{(x(t), y(t)) \mid t \in [a, b]\}$$

defines a plane curve.

Here the restriction that $x$ and $y$ be continuous guarantees us that the curve we end up with will actually be connected. In order to proceed however, we still some additional restrictions.

**Definition 0.1.2 (Smooth Curve).** A plane curve is called smooth if it is defined by functions $x$ and $y$ where $x'$ and $y'$ exist and are continuous on $(a, b)$, and $x'(t)$ and $y'(t)$ are not simultaneously 0 on $(a, b)$.

Here the requirement that the derivatives be continuous ensures that our curve will suffer no immediate changes in direction, and the requirement that the derivatives not be simultaneously 0 will guarantee that the curve does not stop or reverse over itself. With these restrictions in place, we are prepared to calculate the length of smooth curves in a two-dimensional plane.

To begin, we partition the interval $[a, b]$ into a number of subintervals,

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b.$$ 

Over each of these subintervals we will replace the actual curve with a straight line segment. Using the formula for distance in a two-dimensional plane the length of the $i^{th}$ segment will be given by

$$\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}.$$
Now we apply the mean value theorem (for derivatives), noting that there are some points \( \hat{t}_i \) and \( \tilde{t}_i \) in \([t_{i-1}, t_i]\) so that

\[
x(t_i) - x(t_{i-1}) = x'(\hat{t}_i) \cdot (t_i - t_{i-1}) = x'(\hat{t}_i) \cdot \Delta t_i \\
y(t_i) - y(t_{i-1}) = y'(\tilde{t}_i) \cdot (t_i - t_{i-1}) = y'(\tilde{t}_i) \cdot \Delta t_i
\]

From here we find that the length of the \( i \)th line segment is given by

\[
\sqrt{(x'(\hat{t}_i) \cdot \Delta t_i)^2 + (y'(\tilde{t}_i) \cdot \Delta t_i)^2} = \sqrt{(x'(\hat{t}_i))^2 + (y'(\tilde{t}_i))^2} \Delta t_i.
\]

Now to find the approximation of the length using \( n \) segments we simply sum up each of these contributions, to find

\[
\sum_{i=1}^{n} \sqrt{(x'(\hat{t}_i))^2 + (y'(\tilde{t}_i))^2} \Delta t_i.
\]

To make this approximation as accurate as we’d like, we need to consider the limit as the width of each of the subintervals approaches 0. As we do so the fact that above we have two different points \( \hat{t}_i \) and \( \tilde{t}_i \) becomes immaterial, because they are essentially forced to take on the same position as the width of a given interval shrinks to 0. We can see this using the mean value theorem over an interval \([t_{i-1}, t_{i-1} + \Delta t_i]\). As \( \Delta t_i \to 0 \) we have

\[
\lim_{\Delta t_i \to 0} (x(t_i) - x(t_{i-1})) = \lim_{\Delta t_i \to 0} x'(\hat{t}_i) \cdot \Delta t_i = \lim_{\Delta t_i \to 0} x'(t_{i-1}) \cdot \Delta t_i,
\]

because \( x'(t) \) is a continuous function, and in the above limit there is no place for \( \hat{t}_i \) to go except for \( t_{i-1} \). An analogous result holds for \( y'(\tilde{t}_i) \). Thus, we arrive at the following result for the length of a smooth plane curve.

\[
L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt.
\]

From this general result we can very easily move to the special cases of finding the length of a function. If we have a function \( y(x) \), then we can simply treat \( x \) as the parameter, so we have the curve defined by \( y(x) \) and \( x(x) = x \). We find the length of such a curve by

\[
L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx.
\]

Similarly, if we have a function \( x(y) \), we can let \( y \) be the parameter, and

\[
L = \int_a^b \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dx.
\]

**Example 1.** Find the circumference of the circle \( x^2 + y^2 = a^2 \).

**Solution** In order to solve this problem we first must note that the equation of the above circle is given in parameterized form by

\[
x = a \cos(t), \quad y = a \sin(t), \quad t \in [0, 2\pi].
\]

From here we find that

\[
\frac{dx}{dt} = -a \sin(t) \quad \text{and} \quad \frac{dy}{dt} = a \cos(t).
\]

Now we can use the integral for the length of a curve to find

\[
L = \int_0^{2\pi} \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)} dt = \int_0^{2\pi} a dt = a \int_0^{2\pi} dt = 2\pi a.
\]
Example 2. Find the length of the curve \( y = x^{3/2} \) from \( 0 \leq x \leq 2 \).

Solution Here we have \( y \) as a function of \( x \), so we can use one of the alternative forms for finding the length of the curve. We find
\[
\frac{dy}{dx} = \frac{3}{2} x^{1/2},
\]
so
\[
\left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x.
\]
Now evaluating the appropriate integral
\[
L = \int_{0}^{2} \sqrt{1 + \frac{9}{4}x} \, dx,
\]
which we solve through substitution, letting
\[
u = 1 + \frac{9}{4}x,
\]
and
\[
du = \frac{9}{4} \, dx \quad \text{or} \quad 4du = dx.
\]
It follows \( u(0) = 1 \) and \( u(2) = 1 + 9/2 = 11/2 \). Thus,
\[
L = \int_{0}^{2} \sqrt{1 + \frac{9}{4}x} \, dx = \frac{4}{9} \int_{1}^{11/2} \sqrt{u} \, du = \frac{8}{27} u^{3/2} \bigg|_{1}^{11/2} = \frac{8}{27} \left(\left(\frac{11}{2}\right)^{3/2} - 1\right) \approx 3.5255.
\]

Example 3. Find the length of \( y = (x/2)^{2/3} \) from \( x = 0 \) to \( x = 2 \).

Solution When we find
\[
\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \cdot \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{1/3},
\]
we notice that this function grows without bound as \( x \to 0 \). Rather than dealing with improper integrals, we can alternatively represent \( x \) as a function of \( y \), and find
\[
y = \left(\frac{x}{2}\right)^{2/3}
\]
\[
y^{3/2} = \frac{x}{2}
\]
\[
x = 2y^{3/2},
\]
which we have previously seen is differentiable. When \( x = 0 \) we have \( y = 0 \), and when \( x = 2 \) we have \( y = 1 \), so our limits of integration will be for \( y \) from 0 to 1. Calculating the derivative
\[
\frac{dx}{dy} = 2 \left(\frac{3}{2}\right) y^{1/2} = 3y^{1/2},
\]
and
\[
\left(\frac{dx}{dy}\right)^2 = 9y.
\]
Thus,
\[
L = \int_{0}^{1} \sqrt{1 + 9y} \, dy = \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \bigg|_{0}^{1} = \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27.
\]
Example 4. Find the length of \( x = (y^3/6) + 1/(2y) \) from \( y = 2 \) to \( y = 3 \).

Solution We begin by finding the derivative

\[
\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2},
\]

and

\[
\left( \frac{dx}{dy} \right)^2 = \frac{1}{4}(y^4 - 2 + y^{-4}).
\]

Thus,

\[
L = \int_2^3 \sqrt{1 + \frac{1}{4}(y^4 - 2 + y^{-4})} \, dy = \int_2^3 \sqrt{\frac{1}{4}(y^4 + 2 + y^{-4})} \, dy = \frac{1}{2} \int_2^3 \sqrt{(y^2 + y^{-2})^2} \, dy
\]

\[
= \frac{1}{2} \int_2^3 (y^2 + y^{-2}) \, dy = \frac{1}{2} \left[ \frac{y^3}{3} - \frac{1}{y} \right]_2^3 = \frac{1}{2} \left[ \frac{27}{3} - \frac{1}{3} - \left( \frac{8}{3} - \frac{1}{2} \right) \right] = \frac{13}{4}.
\]

The key to solving the above problem was the fact that

\[
1 + \left( \frac{dx}{dy} \right)^2
\]

was a perfect square, so it canceled out with the square root. However, most curves do not work out so nicely. Unfortunately, we cannot evaluate the vast majority of integrals that arise in finding arc length, and most of them cannot be evaluated by hand. This makes numerical methods such as Simpson’s rule particularly appealing for evaluating such integrals.