0.1 Exponential and Power Functions

When we are working with natural numbers, we can think of exponentiation as short-hand notation for multiplication. If we let n be a nonnegative integer, we can intuitively think of a^n as a multiplied by itself n times. For instance,

$$a^3 = a \cdot a \cdot a.$$

This is similar to how multiplication of natural numbers can be viewed as a short-hand notation for addition. For instance,

$$3 \cdot 2 = 3 + 3 = 2 + 2 + 2.$$

Nevertheless, when we extend multiplication to the integers or rational numbers, multiplication stops resembling addition, becoming an operation with its own unique properties. Just imagine trying to represent

$$4 \cdot \frac{1}{2}$$

in terms of addition - it doesn't make sense. Analogously, we can begin with this intuitive notion of exponentiation as stemming from multiplication, but we will see that in extending exponentiation to larger sets of numbers, it will become a distinct entity from multiplication. In order to begin, we will slightly modify our intuitive notion of exponentiation, and work on the nonnegative integers rather than natural numbers (in order to include 0). We can think of a^n as 1 multiplied by a, n times. When we start with this definition we have that

$$a^0 = 1 \quad \text{and} \quad a^1 = 1 \cdot a = a.$$

From this intuitive view it is not difficult to deduce that

$$a^n \cdot a^m = a^{n+m}.$$

For instance, if we let $n = 2$ and $m = 3$, then we are multiplying $a \cdot a$ and $a \cdot a \cdot a$, so the result is five terms of a multiplied together, or $a^5 = a^{2+3}$. A similar argument works for any other values of n or m. Now what if we have something of the form a^m/a^n? Letting $n = 2$ and $m = 3$ again, we have

$$\frac{a^m}{a^n} = \frac{a^3}{a^2} = \frac{a \cdot a \cdot a}{a \cdot a} = a = a^{3-2}.$$

Once again we can make the same argument for any values of n and m. Thus,

$$\frac{a^m}{a^n} = a^{m-n},$$

where we are assuming $m > n$. According to the above rules, if we multiply two base a exponential functions together, the exponents add, and if we divide, the exponent in the denominator is subtracted. If we write the above expression slightly differently, we see

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^{m-n},$$

where we now have written the expression as a product rather than a quotient. When we multiply any base a exponential by $1/a^n$, $1/a^n$ behaves similar to a base a exponential (even though it isn't one) except that n is subtracted from the exponent rather than added to it. From another viewpoint, when we multiply by $1/a^n$, we *add* $-n$ to the exponent. In accordance with our rule for multiplying exponentials, it is natural to *define* negative exponents, so that

$$a^{-n} = \frac{1}{a^n}.$$

This definition of negative exponents is natural because it is completely consistent with our previous rule for multiplying positive exponentials (if we multiply by a negative exponent, we add the negative exponent). Now we have the rule

$$a^n \cdot a^m = a^{n+m} \quad \text{for } n, m \in \mathbb{Z}.$$

We have gained more flexibility by extending exponentiation to all integers, but at the cost of our initial intuitive understanding; the notion of multiplying a number by itself a negative number of times is purely nonsense.

It's possible to extend the notion of exponentiation even further to rational numbers, if we notice that

$$(a^n)^m = a^{nm} \quad \text{for } n, m \in \mathbb{Z}.$$

Just for another simple illustration, think of $(a^2)^3$. Here we have

$$(a^2)^3 = (a \cdot a)^3 = (a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a) = a^6 = a^{2 \cdot 3}.$$

We can extend exponentiation to rational numbers by requiring that rational exponents abide by this same rule. In other words, we should have

$$(a^{1/n})^n = a^{n/n} = a \quad \text{and} \quad (a^n)^{1/n} = a^{n/n} = a.$$

In this way we can *define* $a^{1/n}$ as the number such that when it is raised to the n^{th} power, the result is a. This extended definition of exponentiation is also consistent with our previous rules (although this may be a little difficult to verify).

Since we've already made it to the rational numbers, a reasonable goal would be to try and achieve real exponents. This is indeed possible, but we will need the aid of limits in order to do so, so we will return to this issue at a later time (just as we avoided defining the real numbers earlier). Nevertheless, for the moment we will take for granted that real exponentials work (as do complex exponentials), and that we have the following rules of exponentiation.

Theorem 0.1.1 (Rules of Exponentiation)**.** *Let $a, b, x, y \in \mathbb{R}$. It follows that:*

1. $a^x \cdot a^y = a^{x+y}$
2. $(a^x)^y = a^{xy}$
3. $a^{-x} = \frac{1}{a^x}$
4. $(ab)^x = a^x b^x$
5. $a^1 = a$
6. $a^0 = 1$

Even though we have defined rational and real exponentials, in most cases we do not actually know how to evaluate them yet. For instance, consider

$$2^{1/2} = \sqrt{2}.$$

By definition, $\sqrt{2}$ is the number that when squared is 2. However, right now we have no idea what this number actually is. We will see later on that the answer to this question lies in approximation, so we will not be able to answer this question without the aid of calculus.

Setting the above difficulties aside we can use exponentiation to define polynomial and exponential functions, which will be some of the most basic functions available to us.

Definition 0.1.1 (Power Function)**.** A *power function* is a function of the form $f(x) = x^a$, where $a \in \mathbb{R}$.

Thus, a power function is a function where the *base* of the exponential varies as an input. Very basic examples of power functions include $f(x) = x$ and $f(x) = x^2$. Note that $f(x) = x$ maps the real numbers to the real numbers, where $f(x) = x^2$ maps the real numbers to the nonnegative real numbers. Some power functions are only defined as maps on the real numbers for a domain of nonnegative real numbers, such as

$$f(x) = x^{1/2} = \sqrt{x}.$$

For this function a negative input is not defined, because complex numbers are required to make sense of the square root of a negative number.

Using power functions as our most basic building blocks, we arrive at polynomial functions. One of the most basic ways in which we combine functions is in a *linear combination*.

Definition 0.1.2 (Linear Combination)**.** A *linear combination* of the functions $\{f_1, f_2, \ldots, f_n\}$ is a function

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x),$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Definition 0.1.3 (Polynomial)**.** A *polynomial* is a function that can be written as a linear combination of power functions. Thus, polynomials take on the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

where $a_i \in \mathbb{R}$.

Polynomials provide us with a large class of simple functions to work with. In constrast to an arbitrary function, a polynomial is very well-behaved, and as a result has a number of useful properties. As we delve further into the study of calculus we will use polynomials to approximate other functions, and eventually be able to represent much more complicated functions as infinite sums of power functions. Such representations are called *power series*.

In contrast to power functions, *exponential functions* are functions where the exponent varies as an input.

Definition 0.1.4 (Exponential Function)**.** An *exponential function* is a function

$$f : \mathbb{R} \to \mathbb{R}^+ \text{ (positive real numbers)}, \quad f(x) = a^x,\ a \in \{x \in \mathbb{R} \mid x > 0, x \neq 1\}.$$

Note that when we are talking about exponential functions we are only interested in exponentials with base $a > 0$. We are not interested in $a = 1$, because it is simply a constant function. Since this constant function behaves differently from the rest of the exponential functions we will deal with, we simply exclude it from the list of exponential functions. All of the exponential functions have a domain of $\mathbb{R}$ and a range of $\mathbb{R}^+$ (positive real numbers). This means that the output of an exponential function is always positive. In fact, exponential functions are strictly increasing, which means for each exponential there is a corresponding inverse function (see theorem **??**). These inverse functions are called *logarithms*.

Definition 0.1.5 (Logarithmic Function)**.** A *logarithmic function* is a function

$$f : \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = \log_a(x),\ a \in \{x \in \mathbb{R} \mid x > 0, x \neq 1\},$$

where $\log_a(x)$ is the inverse function of a^x.

Since exponentials and logarithms are inverses, we have

$$\log_a(a^x) = x = a^{\log_a(x)}$$

for all $a > 0, a \neq 1$. By virtue of this inverse relationship, logarithms inherit a number of useful properties from exponentials. For instance, by using the relationship

$$a^x a^y = a^{x+y}$$

we can deduce

$$\log_a(a^x a^y) = \log_a(a^{x+y}) = x + y = \log_a(a^x) + \log_a(a^y).$$

Above we simply use the property that $\log_a(a^x) = x$, in order to move from step 2 to 3, and step 3 to 4.

We can also deduce a rule for the logarithm of a product, noting that the exponential function a^x has a range of the entire positive real numbers $\mathbb{R}^+$. In other words, for any $x \in \mathbb{R}^+$ there is some $b \in \mathbb{R}$ so that

$$x = a^b.$$

Similarly, for any positive y, we can write $y = a^c$, for some c. As a result,

$$\log_a(xy) = \log_a(a^b a^c) = \log_a(a^b) + \log_a(a^c) = \log_a(x) + \log_a(y).$$

Thus, the logarithm of a product of two numbers is the sum of the logarithms. The full table of properties of logarithms follows.

Theorem 0.1.2 (Rules of Logarithms). *Let $a, x, y \in \mathbb{R}^+$. It follows that:*

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a(x^y) = y \log_a(x)$
3. $\log_a(1) = 0$
4. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
5. $\log_a(\frac{1}{y}) = -\log_a(y)$

Property 2 above follows if we write $m = \log_a(x)$, where $m \in \mathbb{R}$. Using the inverse relationship of exponentials and logarithms we also know $x = a^m$. Thus,

$$x^y = (a^m)^y = a^{my}.$$

Taking the base a logarithm of each side of the above equation,

$$\log_a(x^y) = \log_a(a^{my}) = my = ym = y \log_a(x).$$

The third property is simply that $a^0 = 1$ for all a. Properties 4 and 5 are written only for convenience - they follow immediately from the previous three properties. Can you see why?

Given that for any value of $a > 0, a \neq 1$ we have both an exponential and corresponding logarithmic function, we have access to a plethora of functions through exponentials and logarithms. However, as we noted previously, we are currently unable to evaluate exponentials for all but a very small set of numbers. Similarly, we have difficulty in actually finding the values of logarithmic functions. Thus, even though we have defined this large class of functions, and have found that they have a number of useful properties, we still cannot actually evaluate them in most situations.

It turns out that the solution to this problem lies in polynomial functions. Given the difficulty of evaluating exponential functions, we can instead turn to approximating their values. Using the power of calculus (through the limit) we will actually be able to represent exponential functions as a sum of an infinite number of power functions (with natural-number exponents). Since it is relatively easy to evaluate power functions, this will give us a means of accessing these much more elusive exponential functions. We will deal with logarithmic functions in a slightly different way, but calculus will once again be essential.

Although exponential and logarithmic functions define infinite classes of functions, we are really only interested in a single function from each of these classes. The exponential function we will be interested in is the base e exponential, where e is a specific irrational number, defined by the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Because this function is used so much more often than other exponential functions, it is often referred to as *the* exponential function. Do not worry about understanding the above notation - as we delve into the study of limits and sequences it will begin to make sense. Corresponding to the base e exponential, we are interested in $\log_e(x)$, the *natural logarithm*, which is often written as $\ln(x)$. The graphs of these two functions are given in figure 1.

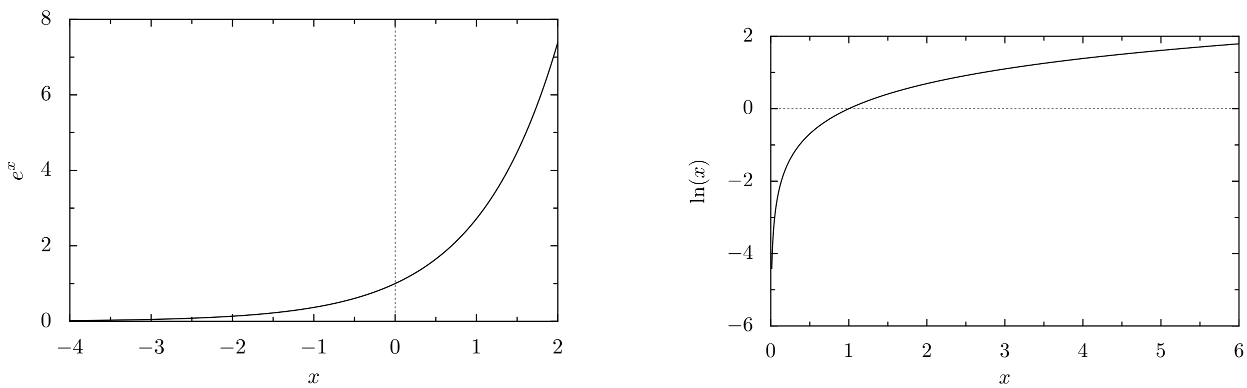

Figure 1: Exponential and Logarithmic Functions

While the exponential function and natural logarithm are really the only two functions of interest for us, the base 10 logarithm is sometimes encountered as well. The base 10 logarithm often written $\log(x)$ for short, which was widely used to simplify calculations before the advent of computers. The base 10 logarithmic is a part of the definition of a decibel, so it is encountered in fields such as telecommunications and acoustics.