Continuity

In the most basic sense, a continuous function is one that can be drawn in one continuous stroke, with no lifting of a pen or pencil. Continuous functions have many useful properties, such as the intermediate value property: if a continuous function takes on two values, it must also take on all values inbetween; if it were to skip any value inbetween, then we wouldn't be able to draw the function in a single stroke, because we would need to lift our pencil to skip over that value. However, since there are many functions which we may not know how to draw, or at the very least do not wish to draw due to their complexity, this is not the most useful definition for continuity. If we can find a different way to characterize this concept, then we may be able to deduce that a function has certain properties, such as that stated above, even though we don't know how to draw the function.

The reason we would need to lift our pencil in drawing a function is because it either has a jump or hole in its graph. We call such points discontinuities, because a function fails to be continuous by virtue of their existence. A continuous function is one that has no discontinuties in its domain, or in other words, is continuous at all points in its domain. This leads us to the notion of continuity at a point.

Definition: Continuity at a Point
A function f is continuous at an interior point x_0 of its domain if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

A function f is continuous at a left endpoint a or a right endpoint b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b)$$

respectively. If a function is not continuous at x_0, we say it is discontinuous at x_0.

We say that a function is left-continuous at x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$ and that it is right-continuous if $\lim_{x \to x_0^+} f(x) = f(x_0)$. Thus, a function is continuous at a left-endpoint if and only if it is left-continuous at that point, and analagously for right-endpoints. A function is continous at an interior point of its domain if and only if it is both left and right-continuous at that point. The condition for continuity at a point actually encapsulates three criteria, so in order for a function to be continuous at a point x_0 we must have that

1. $f(x_0)$ is defined (x_0 is in the domain of $f(x)$)
2. $\lim_{x \to x_0} f(x)$ exists
3. $\lim_{x \to x_0} f(x) = f(x_0)$

We extend the local property of continuity at a point into a global property with the following definition.

Definition: Continuity
A function f is continuous if it is continuous at every point in its domain.

It cannot be overemphasized that the continuity of a function depends on the domain of the function. A general function may be continuous on some domains, and may not be continuous on some others. Unless otherwise specified, we will generally consider the continuity of functions defined on the real numbers, denoted by $\mathbb{R}$.

It is also noteworthy that there are some subtle differences between our formal definition of continuity and the intuitive notion of being able to draw the function without lifting our pencil. Consider the function

$$f(x) = \frac{1}{x}$$

on $\mathbb{R}$. We have shown previously that $\lim_{x\to 0} f(x)$ does not exist, meaning that $x = 0$ is a point of discontinuity (it also turns out that $f(0)$ is not defined, but we could just as easily define it as any number L, and this analysis would still hold true). This is consistent with the intuitive notion, because when we approach $x = 0$, we need to lift our pencil to continue drawing the function. However, we can consider the same function on the domain $(-\infty, 0) \cup (0, \infty)$ which is all real numbers excluding 0. It turns out that for every point in this domain the function $f(x)$ is continuous, so $f(x)$ is continuous on this domain, but we are still unable to draw the function without lifting our pencil. This interesting phenomenon occurs because our function is not defined on an interval. When we consider a function defined on an interval, the intuitive notion of continuity is consistent with the formal definition (which still does not mean that arguing based on the intuitive notion is mathematically rigorous!). It also turns out that the intermediate value property only holds for continuous functions defined on an interval (because otherwise we could have a jump in our *continuous* function and our previous argument would be false).

Just as we had rules for combining limits, we can also combine continuous functions to create continuous functions.

Properties of Continuous Functions
Suppose $g(x)$ is continuous at x_0

1. If $f(x)$ is continuous at x_0, then $f(x) + g(x)$ is continuous at x_0
2. If $f(x)$ is continuous at x_0, then $f(x) \cdot g(x)$ is continuous at x_0
3. If $f(x)$ is continuous at x_0, $g(x_0) \neq 0$, then $\frac{f(x)}{g(x)}$ is continuous at x_0
4. If $f(x)$ is continuous at $g(x_0)$, then $f \circ g(x)$ is continuous at x_0

As before, these rules will only be useful if we have some knowledge about basic continuous functions. We will be able to use these functions as building blocks to verify the continuity of more complex functions. We already know that for any polynomial function, the limit is the same as the value of the polynomial at every point, so it follows that all polynomial functions are continuous on $\mathbb{R}$. Sine, cosine, and the exponential function are also continuous on all of $\mathbb{R}$. In fact, most functions that we encounter are continuous, as long as we restrict them to the proper domain. For instance, $\ln(x)$ is continuous for $x > 0$, which is the entire domain over which it is defined. It follows from the continuity of these basic functions, and the above properties, that much more complicated functions such as

$$f(x) = \frac{x\sin(x)}{x^2+2}$$

are continuous (when combining functions in this way one must be careful to avoid division by 0, which we do here because $(x^2+2) > 0$ for all x).

Generally we encounter two types of discontinuities - point and jump discontinuities. A point discontinunity occurs when we have a function that would otherwise be continuous, but it is either not defined at a single point, or is defined in such a way that the limit and the function value at that point are not equal. In either case, if we were to properly redefine the function at that point, we could make the function continuous there. For this reason, this type of discontinuity is sometimes called a removable discontinuity, because we can remove the discontinuity by redefining the function at that point. A good example of a function with this type of discontinuity is

$$f(x) = \frac{x^2+x-6}{x-2} = \frac{(x-2)(x+3)}{x-2}$$

This function is not defined for $x = 2$, but if $x \neq 2$ then we simply have that

$$f(x) = x + 3$$

Thus, if we define $f(2) = 5$, we can remove the discontinuity, making the function continuous over $\mathbb{R}$, and equivalent to the function $f(x) = x + 2$.

A jump discontinuity occurs when we have two pieces of a function that do not line up; for instance, a piecewise defined function where the two pieces of the function are not connected. It also occurs in a function such as

$$f(x) = \frac{|x|}{x}$$

because there are two pieces of the function separated at $x = 0$. No matter how we define $f(0)$, we cannot remove this discontinuity. Jump discontinuities also occurs when the limit as a function approaches a point is ∞ or $-\infty$ (the function value is not defined at that point, so it is discontinuous at that point). A fraction is definitely discontinuous at any point where its denominator would be 0, as it is not defined there, but depending on the fraction this discontinuity may or may not be removable (it may be either a point or jump discontinuity). Finally, we may encounter a function that is discontinuous everywhere, such as

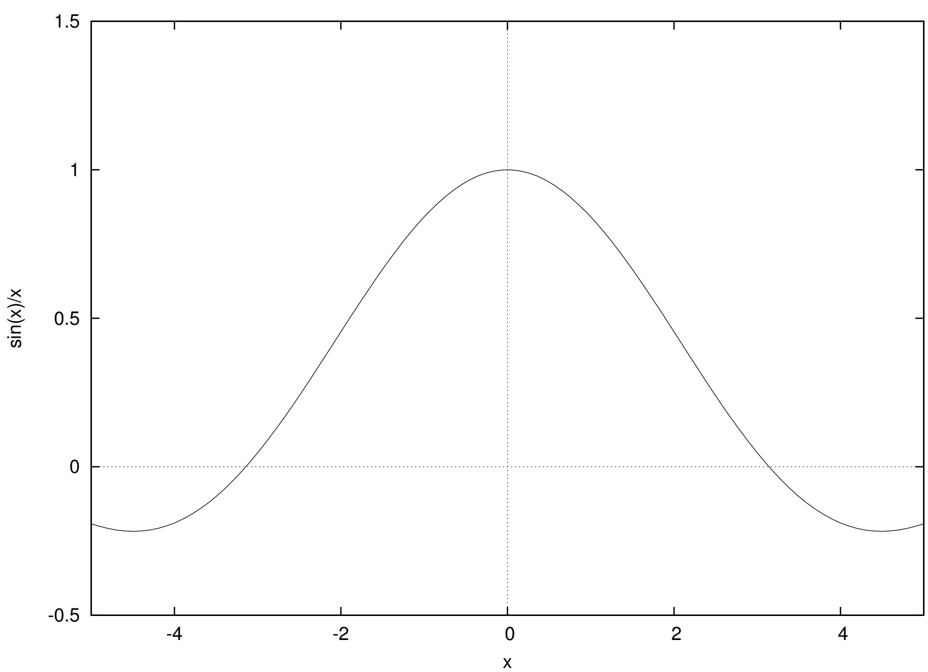
$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

The definition of the above function is that it is 0 for rational numbers, and 1 for irrational numbers. The graph of this function is so broken it doesn't make much sense to call the discontinuities point or jump discontinuities, but it should be clear that we cannot remove any of the discontinuities by simply redefining a single point!

Example 1 Is $\sin(\frac{1}{x})$ continuous over the real numbers? If not, does it have any discontinuities which can be redefined to make it continuous over the real numbers?
Solution Since there is a division by 0 if $x = 0$, clearly this function is not defined at $x = 0$, so this function is discontinuous over a domain of real numbers. As $x \to 0$ the function oscillates faster and faster between 1 and -1, so it does not approach any single value. Thus, there is no way that we could define $f(0)$ in order to make the function continuous over the domain of real numbers. However, if the domain is restricted to the real numbers minus 0 this function is continuous.

Example 2 Is $x \cdot \sin(\frac{1}{x})$ continuous over the real numbers? If not, does it have any discontinuities which can be redefined to make it continuous over the real numbers?
Solution Since there is a division by 0 if $x = 0$, clearly this function is not defined at $x = 0$, so this function is discontinuous over a domain of real numbers. However, as $x \to 0$ despite the fact that the sine function oscillates faster and faster between 1 and -1, it is multiplied by x, so it approaches 0. Thus, if we define $f(0) = 0$, then it will be a continuous function over the domain of reals. Without that redefinition, it is continuous for the domain of all real numbers minus 0.

Example 3 Is $\frac{\sin(x)}{x}$ continuous over the real numbers? If not, does it have any discontinuities which can be redefined to make it continuous over the real numbers?
Solution Since there is a division by 0 if $x = 0$, clearly this function is not defined at $x = 0$, so this function is discontinuous over a domain of real numbers. In order to get a better idea of the behavior of this function as $x \to 0$, it is helpful to look at a graph of the function.



We can see based on the figure that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Thus, if we define $f(0) = 1$, we can make this function continuous over a domain of $\mathbb{R}$. A consequence of the above limit is that for small x values, $\sin(x)$ essentially behaves like x. This type of approximation appears many places in engineering and physics, so it is a useful fact to remember.

Throughout this discussion we have made several references to an intermediate value property. Formally stated:

Theorem: The Intermediate Value Theorem
Let $f(x)$ be continuous on the closed interval $[a, b]$. For any value y_0 between $f(a)$ and $f(b)$, there exists some $c \in [a, b]$ such that $f(c) = y_0$

In the above theorem we use the statement between $f(a)$ and $f(b)$ because it is not clear which one is larger than the other, if in fact $f(a) \neq f(b)$. If we have $f(a) = f(b)$ then the above statement tells us nothing, because there are no values between $f(a)$ and $f(b)$, and we already know our function takes on the value $f(a) = f(b)$ at the endpoints of the interval. The intermediate value property is particularly useful for finding roots of a function, insofar that if we have a function $f(x)$, and we know that for some a we have $f(a) < 0$ and for some b we have $f(b) > 0$, then there is some c between a and b where $f(c) = 0$.