# Derivatives

We have defined

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

as the derivative of the function $f$ at the point $x_0$, and

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

as the derviative of $f$, which is a new function in its own right. The domain of this function is all points at which the function $f$ is differentiable (at which the above limit exists). Thus, the domain of $f'(x)$ may be smaller than the domain of $f(x)$. If $f(x)$ is differentiable at all points in its domain, it is said simply to be differentiable. We stated previously that a function is not differentiable at any point in its domain where there is a: corner, cusp, vertical tangent, or discontinuity. From this it follows that differentiability at a point implies continuity at that point (because if it were discontinuous it couldn't be differentiable). Lastly, when we evaluate the above limit to find the derivative of $f$, we say that we differentiate $f$.

A powerful concept that occurs frequently in higher level mathematics is the concept of operators. In essence, a function is a mathematical object that defines relationships between numbers: given a numerical input a function returns a numerical output. An operator is a mathematical object defines a relationship between functions: given a function as an input, an operator returns a new function as an output. According to this concept, we can define a differentiation operator, that when applied to a function returns the derivative of that function as an output. If we are differentiating with respect to the variable $x$, we define the differentiation operator as

$$\frac{d}{dx}$$

(pronounced 'dee dee $x$'). In describing the action of this operator we write

$$\frac{d}{dx} f(x) = f'(x)$$

meaning that when the differentiation operator is applied to a function, it returns the derivative of that function as an output. For this reason, it is common to see the notation

$$\frac{df}{dx}$$

to represent the derivative $f'(x)$. This notation is particularly useful when we want to consider functions of multiple variables, where we may take a derivative with respect to just one of the variables (treating the others as constants). We generally write the differentiation operator as

$$\frac{\partial}{\partial x}$$

(pronounced 'del del $x$') when we are considering a function of multiple variables, just to stress the fact that we are differentiation with respect to only a single variable, and considering the others as constants. Without further ado, let us consider some examples of differentiation.

**Example 1** Find the derivative of an arbitrary linear function $f(x) = mx + b$.
**Solution** In truth, we have already solved this problem. By finding the derivative of a function at an arbitrary point, we find the derivative of the function at every point it exists, so we already know that

$$f'(x) = m$$

which is just the slope of the line. This is intuitive because the derivative tells us the slope of a curve, and the slope of a line is constant - just $m$. There are other noteworthy aspects of this example that are worth considering. For instance let us suppose that we have

$$f(x) = 2x + 1 \quad \text{and} \quad g(x) = 2x + 5$$

upon differentiation, we find that

$$f'(x) = 2 \quad \text{and} \quad g'(x) = 2$$

that is, both $f$ and $g$ have exactly the same derivative. Since two different functions can yield the same derivative, we see that when we differentiate a function, we lose some information about the function. Namely, we lose the information about how high (or low) the function is shifted. When we differentiate any arbitrary function the same thing happens - we lose any constant that might have been added to the function. This is because the derivative only tells us how a function is changing, but it doesn't tell us at what level the function starts at. This is a detail that will require careful attention in the future, when we move to reversing the process of differentiation - finding a function from its derivative.

**Example 2** Consider an object falling through the air. Ignoring wind resistance, the height of the object (in meters) is given by

$$y(t) = y_0 - 4.9t^2$$

where $y_0$ is the initial height of the object. The velocity of the object as a function of time is the derivative of its position; find the velocity of the object as a function of time.

**Solution** We will look at the limit of the difference quotient to find $y'(t)$

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \lim_{h \to 0} \frac{y_0 - 4.9(t+h)^2 - (y_0 - 4.9t^2)}{h} = -4.9 \lim_{h \to 0} \frac{t^2 + 2th + h^2 - t^2}{h}$$

simplifying we find

$$y'(t) = -4.9 \lim_{h \to 0} \frac{h(2t+h)}{h} = -4.9 \lim_{h \to 0} 2t + h = 9.8t$$

We can continue on to find acceleration, which is the rate of change of $y'(t)$, denoted by the second derivative of $y$

$$y''(t) = \frac{d}{dt} 9.8t = 9.8$$

because the derivative of a linear function is the slope of the line. This example shows that acceleration due to gravity is a constant (ignoring other forces). Notice that acceleration and velocity do not depend on $y_0$. As the ball falls, the velocity continues to increase due to the constant acceleration. Here we have neglected wind resistance, and in truth after falling for long enough the ball would reach a terminal velocity, where its acceleration is 0.

It can also be useful for us to consider the notion of one-sided derivatives, in complete analogy with one-sided limits. The right-hand derivative of $f$ at $x_0$ is

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

and the analogous left-hand derivative is

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

If we want to consider the derivative of a function at an endpoint of the interval over which it is defined, we need to use a one-sided derivative, because the function is not defined beyond the endpoint. Saying a function is differentiable at an endpoint of its domain means it is one-sided differentiable at that point. We might want to consider $\sqrt{x}$ which is not defined for $x < 0$. In order to look at a derivative at $x = 0$, we use the right-hand derivative

$$\lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$$

so it follows that $\sqrt{x}$ is not differentiable at $x = 0$. It is also noteworthy that we only have a two-sided derivative if both of the one-sided derivatives exist and are equal. The following example illustrates another application of one-sided derivatives.

**Example 3** Let the position of a car be given by:

$$x(t) = \begin{cases} \frac{t^2}{2} & 0 \le t < 1 \\ t - \frac{1}{2} & 1 \le t < 2 \\ \frac{t}{t-1} - \frac{1}{2} & 2 \le t < 4 \\ \frac{16}{9t} + \frac{7}{18} & 4 \le t \end{cases}$$

Find the velocity of the car as a function of $t$. Are there any places where the this velocity function does not make physical sense?

**Solution** On the interiors of these intervals (all points except for the endpoints) we can simply consider the two-sided derivative. On the boundary points between intervals we will need to consider the one-sided derivatives on each side, and see if they are equal.

$$v(t) = \lim_{h\to 0} \frac{(t+h)^2/2 - t^2/2}{h} = \frac{1}{2}\lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} = \frac{1}{2}2t = t$$

which holds for $0 \le t < 1$. We can include $t = 0$ in this interval because the function is not defined for negative time, so we are actually considering the boundary of the entire interval over which the derivative is defined, which means we don't need to match the one sided derivative with anything. On the second interval $1 < t < 2$ we have

$$v(t) = 1$$

because the derivative of a linear function is just the slope of the line. If we look at these limits from both the left and right sides of $t = 1$, we see that they agree, so that we can define $v(1) = 1$. Since the constant term will disappear when we differentiate, to look at the derivative on the third interval it is enough to consider the derivative of $t/(t-1)$.

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{\frac{t+h}{t+h-1} - \frac{t}{t-1}}{h} = \lim_{h\to 0} \frac{(t+h)(t-1) - t(t+h-1)}{h(t+h-1)(t-1)} \\ &= \lim_{h\to 0} \frac{t^2 + th - t - h - t^2 - th + t}{h(t+h-1)(t-1)} = \lim_{h\to 0} \frac{-h}{h(t+h-1)(t-1)} = \frac{-1}{(t-1)^2} \end{aligned}$$

Evaluating

$$\lim_{t\to 2} \frac{-1}{(t-1)^2} = -1$$

we see that the derivatives do not match on both sides of $t = 2$, so this portion of the derivative only holds for $2 < t < 4$ (and the derivative does not exist for $t = 2$). Looking to the fourth interval, once again ignoring the constant

$$v(t) = \lim_{h\to 0} \frac{\frac{16}{9(t+h)} - \frac{16}{9t}}{h} = \frac{16}{9} \lim_{h\to 0} \frac{t - (t+h)}{ht(t+h)} = \frac{16}{9} \lim_{h\to 0} \frac{-h}{ht(t+h)} = -\frac{16}{9t^2}$$

Since both of the one-sided derivatives match as $t \to 4$, $v(4) = -1/9$. Thus, we have $v(t)$ for all points on $[0, \infty)$, except for at $t = 2$, where the derivative does not exist. Physically, an instantaneous change or jump in velocity corresponds to some type of 'infinite' acceleration; such an acceleration is clearly not physically realizable, and corresponds nicely to the point in the graph of the function $x(t)$ where there is a corner.