# Derivatives and Tangent Lines

Having discussed in a great amount of detail what a limit is, we return to our original question: finding the instantaneous rate of change of a function. We previously stated that the average rate of change of a function over an interval $\Delta x$ is given by

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$$

where $x_0$ is the base point of our interval, and the second point of the interval lies at $x_0 + h$ (the above formula is called the difference quotient of $f$ at $x_0$ with increment $h$, because it is a quotient of a difference). It is worth returning to the graphical interpretation of the average rate of change at this point. A line that intersects a curve at two or more points is called a secant line. In finding the average rate of change between two points of a function, we are finding the slope of the secant line that intersects those two points.

To find the instantaneous rate of change, we need to look at the average rate of change in the limit as the length of the interval approaches 0. That is

$$f'(x_0) = \lim_{x \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists. In this situation, the notation $f'(x_0)$ represents the instantaneous rate of change of the function $f(x)$ at the point $x_0$. This is a particularly important quantity in calculus, so we give it a special name - the derivative of $f(x)$ at the point $x_0$. The instantaneous rate of change, or derivative, of $f(x)$ at $x_0$ also has an important graphical interpretation: $f'(x_0)$ is the slope of the line tangent to $f(x)$ at the point $(x_0, f(x_0))$. In this way, we define the line tangent to $f(x)$ at $x_0$ as the line that passes through the point $(x_0, f(x_0))$ with slope $f'(x_0)$. We define the slope of a function $f(x)$ at a point $x_0$ as the slope of the tangent line that passes through $(x_0, f(x_0))$. Now that we have introduced an extroardinary amount of notation, let us try to get a hold on it by working through some examples.

**Example 1** Let $f(x) = x^2$. Find the equation for the secant line passing through $(2, f(2))$ and $(2 + h, f(2 + h)$. Find the equation for the tangent line passing through $(2, f(2))$.
**Solution** In order to find the equation of a line, we need to know either the slope of the line and a single point the line passes through, or two points on the line, from which we can calculuate the slope. With this information we can use the point slope form (for a slope $m$ and point $(x_0, y_0)$)

$$y = m(x - x_0) + y_0$$

Since we know two points through which the secant line passes, we can find the slope of the secant line

$$m = \frac{f(2 + h) - f(2)}{2 + h - 2} = \frac{(2 + h)^2 - 2^2}{h} = \frac{4 + 4h + h^2 - 4}{h} = \frac{h(4 + h)}{h} = 4 + h$$

Now using point-slope form, the above slope, and a single point on the line (it is customary to use the base point of the interval - and there is a good reason for doing so - because we also want to consider the tangent line at that point) we find

$$\begin{aligned} y &= (4+h)(x-2)+2^2 \\ y &= (4+h)x-8-2h+4 \\ y &= (4+h)x-4-2h \end{aligned}$$

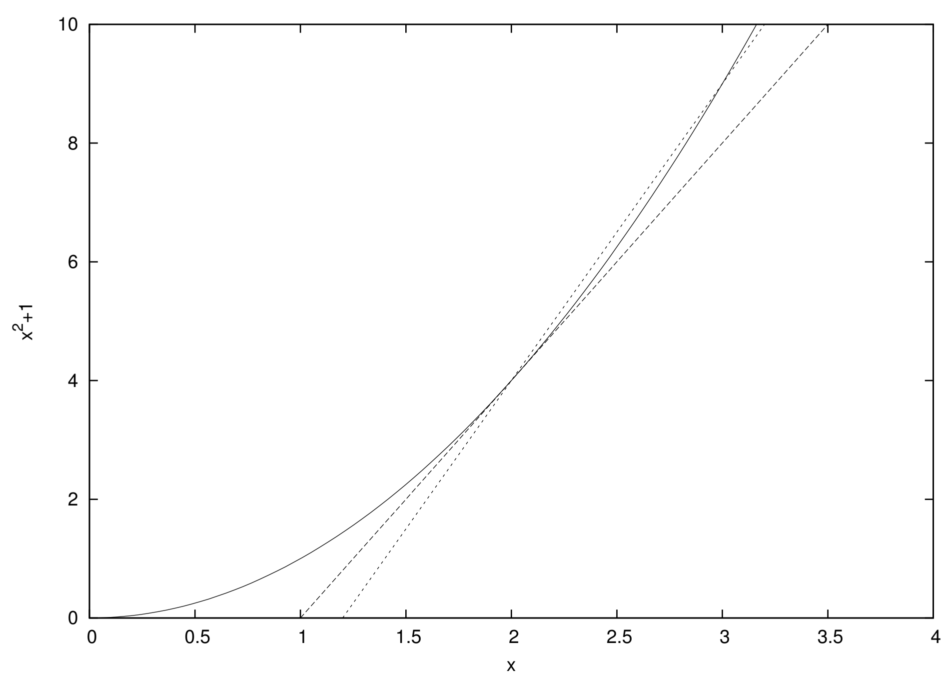
It's useful to notice what happens as we modify our interval, by manipulating $h$. When we extend our interval to the right, by increasing $h$, the slope increases and the $y$-intercept decreases. If we let our interval extend to the left of the base point, by considering negative $h$ values, the slope decreases and the $y$-intercept increases. Now let us perform similar analysis to find the equation of the tangent line (while reusing as much work above as possible).

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{2+h-2} = \lim_{h\to 0}(4+h) = 4$$

Now we utilize point-slope form once again

$$\begin{aligned} y &= f'(2)(x-2)+2^2 \\ y &= 4(x-2)+4 \\ y &= 4x-4 \end{aligned}$$

These two equations are very similar; in fact, if we consider the equation for the secant line in the limit as $h \to 0$, we arrive at the equation of the tangent line. Graphically, we can see that as we decrease $h$, the secant line becomes closer and closer to the tangent line, and in the limit as $h \to 0$, the secant line is the tangent line. Thus, we can interpret the tangent line in a slightly different way; that is, the line tangent to $f(x)$ at $x_0$ is the secant line with base point $x_0$ in the limit as the distance between the two points of the function $f(x)$ the secant line intersects $h \to 0$.



**Example 2** Let $f(x) = mx + b$ be the equation of an arbitrary line. Find the equation of the line tangent to $f(x)$ at an arbitrary point $x_0$.
**Solution** Since the equation of a line passing through a given point is unique, and $f(x)$ is the unique line of slope $m$ that crosses through every point it does, we suspect that the tangent line should have to be exactly the line $f(x)$ (and in fact, so should any secant line passing through the point $x_0$).

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \frac{m(x_0+h) + b - (mx_0 + b)}{h} = \frac{mh}{h} = m$$

so we see that the slope is exactly the same as the line itself.

$$\begin{aligned} y &= f'(x_0)(x - x_0) + f(x_0) \\ y &= mx - mx_0 + mx_0 + b \\ y &= mx + b \end{aligned}$$

Thus, as suspected, the line tangent to a line at any point is just the line itself.

**Example 3** Find the equation of the line tangent to the function $f(x) = x^3$ at $x = 0$.
**Solution** We begin as usual by looking at the limit as $h \to 0$ of the difference quotient

$$f'(0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(0+h)^3 - 0^3}{h} = \lim_{h\to 0} h^2 = 0$$

Since this line passes through the point $(0, 0)$, the point-slope equation is very simple

$$\begin{aligned} y &= 0(x - 0) + 0 \\ y &= 0 \end{aligned}$$

Thus, in this case the tangent line is simply the $x$-axis. It is noteworthy that this line actually intersects the function $f(x) = x^3$, which should dispell the myth that a tangent line *cannot* cross a function.

**Example 4** Find the slope of the line tangent to $f(x) = \sin(x)$ at an arbitrary point $x$.
**Solution** In finding the derivative of a function at an arbitrary point $x$, we define a new function $f'(x)$, which for any $x$ value gives the corresponding value of the derivative at that point. We call this new function $f'(x)$ the derivative of $f$. Proceeding with the difference quotient

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \Big( \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \Big) = \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin h}{h} \end{aligned}$$

We have already seen that the limit

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

and we can actually use this limit to find the value of the other one using the half angle formula $\cos h = 1 - 2\sin^2(h/2)$.

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\sin\theta = -1\cdot 0 = 0$$

where in the above analysis we substituted $h/2 = \theta$ to evaluate the limit. Using these two limits, we find that

$$f'(x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

This is a very interesting result. When we look at the rate of change of the function $\sin(x)$ we find that it is the other sinusoidal function $\cos(x)$. In fact,

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

or in other words, the instantaneous rate of change of $\sin(x)$ at any point $x$ is simply the value of $\sin(x + \pi/2)$, that is, the value of the same function $\pi/2$ to the right of the point of interest. Thus, $\sin(x)$ is a function which has rate of change directly related itself.

If we look at the tangent line at $x = 2\pi/3$ we find that

$$f'(2\pi/3) = \cos(2\pi/3) = -1/2$$

and since

$$f(2\pi/3) = \sin(2\pi/3) = \frac{\sqrt{3}}{2}$$

we find the equation for this tangent line as

$$y = -\frac{1}{2}(x - 2\pi/3) + \frac{\sqrt{3}}{2} = -\frac{1}{2}x + (\frac{\sqrt{3}}{2} + \frac{\pi}{3})$$

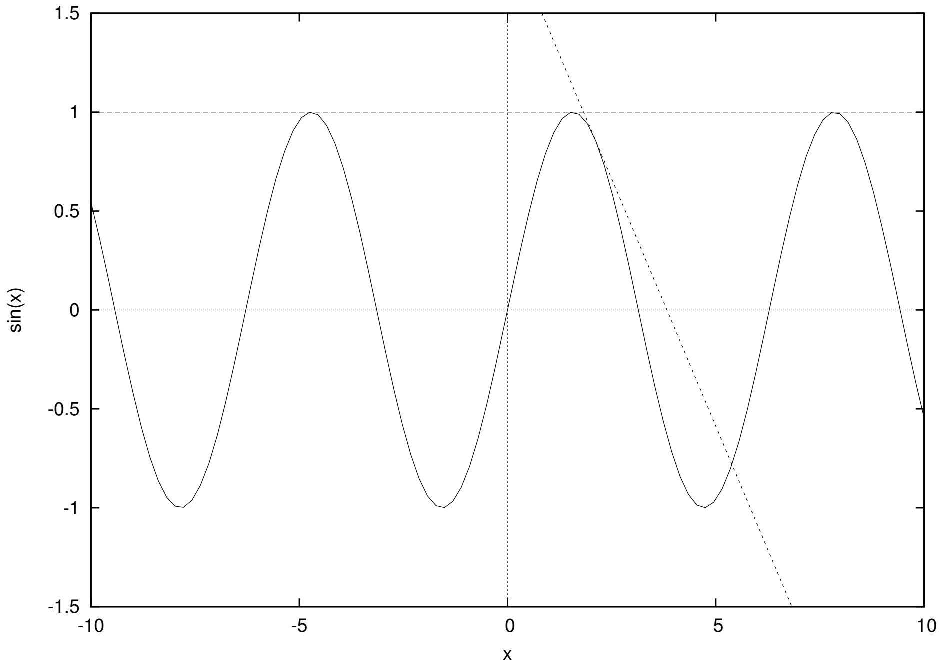
this tangent line will touch the curve at two points, intersecting the curve at one of them. If we consider the tangent line at $x = \pi/2$ we have

$$f'(\pi/2) = 0$$

so we have a horizontal line, with equation

$$y = 1$$

which touches the curve at an infinite number of points. This should dispell the myth that a tangent line can only touch a curve in one place, as we have a tangent line that touches the curve in an infinite number of places.




Since we find the derivative at a point using a limit, it follows that if the limit does not exist, neither will the derivative, nor a nonvertical tangent line (which will be discussed later). We say that a function is not differentiable at a point where its derivative does not exist. A function is not differentiable at any place it has a:

1. *corner.* Consider $f(x) = |x|$ which has a corner at $x = 0$. If we look at the secant lines in the limit as $h \to 0$, we see that from the left side the slope of the tangent line is approaching $-1$, and from the right side the slope is approaching 1. Since these limits do not agree, the derivative and thus tangent line do not exist at $x = 0$ (a similar analysis will hold for a corner of any function).

2. *a cusp.* Consider the function $f(x) = \sqrt{|x|}$ at the point $x = 0$. From the right side the secant lines approach a vertical tangent line with slope $\infty$ and from the left side with slope $-\infty$. Since the limit is increasing or decreasing without bound, it does not exist, so $f(x)$ is not differentiable at $x = 0$, the point where it has a cusp.

3. *vertical tangent.* If the limit of the difference quotient fails to exist because it approaches $\infty$ or $-\infty$ from both sides, we say that the function has a vertical tangent at the point of interest (and since $\pm\infty$ are not numbers, the derivative does not exist). An example of function with a vertical tangent is $(2 - x)^{1/5}$.

4. *a discontinuity.* If we have a jump discontinuity the secant lines will behave like when we have a cusp. A function with a jump discontinuity will have different behavior on both sides of the point of interest. For instance, $f(x) = |x|/x$, $f(0) = 1$ has a jump discontinuity at $x = 0$. The secant lines approach a horizontal tangent from the right, and vertical from the left. Thus, the derivative does not exist at this point (nor does a tangent line).