Derivatives of Products and Quotients

We have already seen that the derivative of a sum of functions is the sum of the derivatives, just as we saw with limits. We might be quick to jump to the conclusion that the derivative of a product of functions is also the product of the derivatives; this however, would be an incorrect conclusion.

<table>
<thead>
<tr>
<th>Product Rule for Derivatives</th>
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<td>If $f(x)$ and $g(x)$ are both differentiable, then $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{dg}{dx} + g(x) \cdot \frac{df}{dx} = f(x) \cdot g'(x) + g(x) \cdot f'(x)$</td>
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Thus, we have two terms, with each term containing the derivative of one of the two functions. The constant product rule is nothing but a special case of the product rule, where one of the functions is a constant. For such a function, its derivative is zero, so a single term remains, which is the derivative of the other function times the constant. Consider the following examples.

Example 1 Find the derivative of $p(x) = (x - 1)(x + 1)$.
Solution Let $f(x) = x - 1$, and $g(x) = x + 1$. Since these are both linear functions with slope 1, it follows that $f'(x) = 1 = g'(x)$. Now we can apply the product rule

$$p'(x) = f(x)g'(x) + g(x)f'(x) = (x - 1) \cdot 1 + (x + 1) \cdot 1 = 2x$$

In this situation, we can also multiply the two polynomials out, and find that

$$p(x) = x^2 - 1$$

Applying the sum and product rules as we did in the last section we find

$$p'(x) = 2x + 0 = 2x$$

which is consistent with the product rule.

Example 2 Suppose the volume of a plant is described by

$$V(t) = 100 + 12t$$

where $t$ is measured in days, and $V$ is measured in cm$^3$. Let the density of the plant decrease according to

$$\rho(t) = 0.8 - 0.05t$$

where $\rho$ is measured in grams per cm$^3$. Describe the change of plant’s mass over time.
Solution First recall that mass is related to volume and density by

$$M(t) = \rho(t)V(t)$$

Now we can apply the product rule to find the rate of change of the mass

$$M'(t) = \rho(t)V'(t) + \rho'(t)V(t) = (0.8 - 0.05t)12 + (100 + 12t)(-0.05) = 9.6 - 0.6t - 5 - 0.6t = 4.6 - 1.2t$$
Looking at the rate of change of mass as a function of time, we can see that initially the mass is increasing, but the rate it is increasing at decreases with time. Eventually, when \( t > \frac{23}{6} \), the derivative becomes negative, so the mass begins to decrease.

Similar to the product rule, we have a result that will allow us to calculate the derivative of a quotient.

**Quotient Rule for Derivatives**

If \( u(x) \) and \( v(x) \) are both differentiable, and \( v(x) \neq 0 \) then

\[
\frac{d}{dx} \frac{u(x)}{v(x)} = \frac{v(x) \frac{du}{dx} - u(x) \frac{dv}{dx}}{v(x)^2}
\]

This rule can be remembered as “low ‘d’ high minus high ‘d’ low,” which represents the numerator in the derivative of a quotient. ‘High’ is the numerator of the quotient, where ‘low’ is the denominator. Finally, ‘d’ represents the derivative of.

**Example 3** Find the derivative of \( \frac{x^2 + 1}{x - 1} \).

**Solution** Let \( u(x) = x^2 + 1 \), and \( v(x) = x - 1 \). It follows that \( u'(x) = 2x \) and \( v'(x) = 1 \). Now we can use the quotient rule

\[
\frac{d}{dx} \frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2} = \frac{(x - 1) \cdot 2x - (x^2 + 1) \cdot 1}{(x - 1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x - 1)^2} = \frac{x^2 - 2x - 1}{(x - 1)^2}
\]

**Example 4** Find the derivative of the sinc function, which is commonly encountered in signal processing. Note that

\[
sinc(x) = \frac{\sin(x)}{x}
\]

**Solution** We will use the quotient rule and our knowledge that the derivative of the sine function is cosine to find

\[
\frac{d}{dx} sinc(x) = \frac{x \cos(x) - \sin(x)}{x^2}
\]

The set of Hill functions is an important family of functions in the study of biology, used to describe stimulus response. Hill functions take on the general form

\[
h(x) = \frac{x^n}{1 + x^n}
\]

where \( n \in \mathbb{Z}^+ \) (is a positive integer 1, 2, \ldots).

**Example 5** Calculate the derivative of the Hill functions for \( n = 1 \) and \( n = 2 \).

**Solution** We will begin with \( n = 1 \). Let \( u(x) = x \) and \( v(x) = 1 + x \). It follows that \( u'(x) = v'(x) = 1 \). Using the quotient rule we find

\[
\frac{d}{dx} h = \frac{(1 + x) \cdot 1 - x \cdot 1}{(x + 1)^2} = \frac{1}{(x + 1)^2}
\]

This derivative is always positive for \( x > 0 \), and \( h'(0) = 1 \).
Now let $n = 1$, so $u(x) = x^2$ and $v(x) = 1 + x^2$. It follows that $u'(x) = v'(x) = 2x$. From the quotient rule we find

$$\frac{dh}{dx} = \frac{(1 + x^2) \cdot 2x - x^2 \cdot 2x}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$$

This derivative is also always positive for $x > 0$, but has the value $h'(0) = 0$. 