# Derivatives of Trigonometric Functions

Before discussing derivatives of trigonmetric functions, we should establish a few important identities. First of all, recall that the trigonometric functions are defined in terms of the unit circle. Namely, if we draw a ray at a given angle $\theta$, the point at which the ray intersects the unit circle will be given by $(\cos(\theta), \sin(\theta))$. That is, $\cos(\theta)$ is the $x$-coordinate of the point, and $\sin(\theta)$ is the $y$-coordinate. If we connect this point to the $x$-axis, we can see there is an inscribed triangle, with base $\cos(\theta)$, height $\sin(\theta)$, and hypotenuse 1. Using the pythagorean theorem that the sum of the squares of the other two sides is the square of the hypotenuse, we find that

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

This is perhaps the most commonly used and most useful of the trigonometric identities. If we divide both sides of the equation by $\cos^2(\theta)$ we find

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

If we divide both sides of the equation by $\sin^2(\theta)$ we find

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Keeping these identities in mind, we will look at the derivatives of the trigonometric functions.

We have already seen that the derivative of the sine function is the cosine function. Through a very similar we can find that the derivative of the cosine function is the negative sine function. Thus,

$$\frac{d}{dx}\sin(x) = \cos(x)$$

and

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

It is useful to look at the graph of a function and its derivative together, to see just how much information is contained in the derivative. Notice that at the peaks of the sine function, the cosine function is 0. This is because the line tangent to the sine function is horizontal at these points. At the peaks of the cosine function (the derivative of sine) the sine function crosses the x-axis - these are the points where the sine function has the greatest slope, or is changing the most rapidly. The graphs of these trigonometric functions also give us a clue as to which derivative contains the negative sign. At $x = 0$, $\sin(x)$ is increasing, and $\cos(x)$ is positive, so it makes sense that the derivative is $+\cos(x)$. On the other hand, just after $x = 0$, $\cos(x)$ is decreasing, and $\sin(x)$ is positive, so the derivative must be $-\sin(x)$.

**Example 1** Find all derivatives of $\sin(x)$.
**Solution** Since we know $\cos(x)$ is the derivative of $\sin(x)$, if we can complete the above task, then we will also have all derivatives of $\cos(x)$.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

gives us the first derivative of the sine function.

$$\frac{d^2}{dx^2}\sin(x) = \frac{d}{dx}\cos(x) = -\sin(x)$$

gives us the second derivative. Also

$$\frac{d^3}{dx^3}\sin(x) = -\frac{d}{dx}\sin(x) = -\cos(x)$$

Finally

$$\frac{d^4}{dx^4}\sin(x) = -\frac{d}{dx}\cos(x) = \sin(x)$$

Now we can see that the fourth derivative of $\sin(x)$ is $\sin(x)$, so we can easily enough find any derivative of the sine function as follows. Suppose we want to find the $n^{th}$ derivative of sine. All we need to do is divide $n$ by 4, and look at the remainder $r$. If we take the $r^{th}$ derivative of sine, it will be exactly the same as taking the $n^{th}$ derivative, as every four derivatives will simply return us to the original result of the sine function. Applying this principle, we find that the $17^{th}$ derivative of the sine function is equal to the $1^{st}$ derivative, so

$$\frac{d^{17}}{dx^{17}}\sin(x) = \frac{d}{dx}\sin(x) = \cos(x)$$

The derivatives of $\cos(x)$ have the same behavior, repeating every cycle of 4. The $n^{th}$ derivative of cosine is the $(n+1)^{th}$ derivative of sine, as cosine is the first derivative of sine.

Knowledge of the derivatives of sine and cosine allows us to find the derivatives of all other trigonometric functions using the quotient rule. Recall the following identities:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

**Example 2** Find the derivatives of $\tan(x)$, $\cot(x)$, $\csc(x)$, and $\sec(x)$.
**Solution** We can find all of the above derivatives using the quotient rule and the derivatives of sine and cosine. Starting with tangent

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)\cos(x) - \sin(x)(-\cos(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

and moving to cotangent

$$\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = \frac{\sin(x)(-\cos(x)) - \cos(x)\sin(x)}{\sin^2(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

next cosecant

$$\frac{d}{dx}\csc(x) = \frac{d}{dx}\frac{1}{\sin(x)} = \frac{\sin(x)\cdot 0 - 1\cdot\cos(x)}{\sin(x)^2} = -\frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)} = -\cot(x)\csc(x)$$

and finally secant

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\frac{1}{\cos(x)} = \frac{0 - (-\sin(x))}{\cos^2(x)} = \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

**Example 3** Find the derivative of $x\cdot\cos(x) + x^2$.
**Solution** To find this derivative, we will utilitize the sum and product rules.

$$\frac{d}{dx}(x\cdot\cos(x) + x^2) = \cos(x) + x\cdot(-\sin(x)) + 2x = \cos(x) + x(2 - \sin(x))$$

**Example 4** Find the derivative of $\cos(x)\tan(x)$.
**Solution** Applying the product rule

$$\frac{d}{dx}\cos(x)\tan(x) = -\sin(x)\tan(x) + \cos(x)\sec^2(x) = \frac{1}{\cos(x)}(1-\sin^2(x)) = \frac{\cos^2(x)}{\cos(x)} = \cos(x)$$

of course we should find the above result, because

$$\cos(x)\tan(x) = \cos(x)\frac{\sin(x)}{\cos(x)} = \sin(x)$$

and the derivative of the sine function is the cosine function. It is useful to check if a product or quotient of trigonometric functions can be simplified; afterall, all of the trigonometric functions are defined directly in terms of sine and cosine.

We have found that the derivatives of the trigonmetric functions exist at all points in their domain. For instance, $\tan(x)$ is differentiable for all $x \in \mathbb{R}$ with $x \neq \pi/2 + 2n\pi$ (the points where cosine is 0). It follows immediately that $\tan(x)$ must be continuous at all of these points (because a discontinuity would preclude differentiability). Using this information, we can easily evaluate limits involving trigonometric functions.

**Example 5** Evaluate $\lim_{x\to 0} \frac{\sqrt{2+\sec(x)}}{\cos(\pi - \tan(x)}$
**Solution** Since we are looking at sums, quotients, and a composition of functions which are continuous at $x = 0$, we can simply plug in $x = 0$ to evaluate the limit

$$\lim_{x\to 0} \frac{\sqrt{2+\sec(x)}}{\cos(\pi - \tan(x)} = \frac{\sqrt{2+\sec(0)}}{\cos(\pi - \tan(0))} = \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

**Example 6** Evaluate $\lim_{x\to 1} x\sec(\pi x/2)$
**Solution** In this example,

$$\sec(\pi/2) = \frac{1}{\cos(\pi/2)}$$

which is not defined. Thus, we cannot simply plug in for the limit. If we look as $x \to 1$, we see that $\cos(\pi x/2) \to 0$, which implies the fraction goes to $\pm\infty$, depending on which side we look at the limit from. Thus, the limit does not exist.

The sine and cosine functions occur in countless applications throughout the physical sciences. In electric circuits, AC currents are described by sine and cosine functions, the so-called sinusoidal functions. Another important example is the simple harmonic oscillator. The simple harmonic oscillator (SHO) is encountered often in physics, because many physical phenomena behave in an extremely similar fashion: a weight on a frictionless spring, the motion of a pendulum, an LC circuit without resistance, and even the quantum mechanical harmonic oscillator. We will focus on the mechanical simple harmonic oscillator - a weight on a frictionless spring. Imagine a spring which is protruding sideways, with a weight resting on a frictionless track. There is an equilibrium position for the spring, at which it is not too compressed or stretched. When brought out of equilibrium the spring exerts a restoring force, which is proportional to the displacement, and tends to bring the system back to the equilibrium position. If we ignore the dampening force, friction, the position of the spring is given by a sinusoid function, either cosine or sine.

**Example 6** Let the position of a mass on a spring be given by $x(t) = 5\cos(t)$. Find the velocity and acceleration. What can be said about the motion of the simple harmonic oscillator?
**Solution** To find the velocity and acceleration we differentiate twice, finding

$$v(t) = \frac{dx}{dt} = -5\sin(t)$$

and

$$a(t) = \frac{dv}{dt} = -5\cos(t)$$

We make the following observations

1. Because the position of the mass is given by $5\cos(t)$, we see that the mass oscillates between -5 and 5.

2. The acceleration is exactly opposite to the position of the object. This is consistent with the notion of the restoring force. When the mass is out of equilibrium, there is a force exerted by the spring (resulting in an acceleration) which pulls or pushes the mass back to the equilibrium position. The acceleration is 0 (and so is the force exerted by the spring) only when the position of the mass is equilibrium.

3. The peaks of the velocity function correspond to the zeroes of the position and acceleration functions, and conversely, the zeroes of the velocity function correspond to the peaks of the position and acceleration functions. This result is of particular physical significance, and corresponds to conservation of energy in a physical system. There are two forms of energy in the system - kinetic energy in the motion of the spring, and potential energy in the coils of the spring. As is characteristic of an oscillating system, oscillations occur in correspondence with the exchange of energy between these two forms. When the spring is in its equilibrium position, the velocity is at its maximum - there is no potential energy in the spring, and all energy is kinetic in the motion of the spring. When the position of the mass is at a peak, the velocity is zero - all the energy is stored potential within the coils of the spring, and there is no kinetic energy because the velocity is zero. Compare this situation to an oscillating LC circuit, where energy is exchanged between a capacitor and inductor. In this situation, the peaks and zeroes of the oscillations correspond to where either all of the energy is stored in the charged capacitor, or in the magnetic field generated by the current flowing through the inductor. A key point to remember is that oscillations generally correspond to the exchange of energy between two (or more) forms.