Global and Local Extrema

Using the power of calculus, we can draw quite accurate sketches of a given function using a limited amount of information. In order to graph, and reason visually about functions, we will need to: identify the function's extrema, find intervals over which the function is increasing and decreasing, and determine the concavity of the function. We can accomplish the first of these two tasks using the first derivative, and the last using the second derivative. We will begin with finding the extrema, or extreme values of a given function. We should be clear that the focus of our attention is on continuous functions, and that most of these methods will be unapplicable for discontinuous functions.

Extrema are the extreme values of a function - the places where it reaches its minimum and maximum values. That is, extrema are the points of a function where it is the largest and the smallet. We can identify two types of extrema - local and global. Global extrema are the largest and smallest values that a function takes on over its entire domain, and local extrema are extrema which occur in a specific neighborhood of the function. In both the local and global cases, it is important to be cognizant of the domain over which the function is defined. That which is an extremum on one domain may very well not be over a new domain, and vice versa. Before delving further, let us give the formal definitions of these various extrema.

Global Extrema
The function f defined on the domain D has a **global maximum** at $c \in D$ if

$$f(x) \leq f(c) \quad \text{for all } x \in D$$

We say that f has a **global minimum** at $c \in D$ if

$$f(x) \geq f(c) \quad \text{for all } x \in D$$

Local Extrema
The function f defined on the domain D has a **local maximum** at $c \in D$ if

$$f(x) \leq f(c) \quad \text{for all } x \text{ in some open interval centered at } c$$

We say that f has a **local minimum** at $c \in D$ if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in some open interval centered at } c$$

If c is an interior point of D we require that the interval extend on both sides of c; if c is at an endpoint of the domain, we only require that the above inequalities hold on the portion of the interval that is in the domain D.

Note that in the above definitions for local extrema, the interval we find may be quite small. Nevertheless, if we can find a single interval over which the above inequalities hold, then they will also hold for any smaller interval.

Although the above definitions give us an idea of what we are trying to find, they give us no clue as to how to find it! Since any interval of finite length contains an infinite number of points, there is no way we can test every point to look for extrema. Instead, we will need to be more clever with the places we look for extrema. Once again we emphasize that we are interested in identifying the extrema of continuous functions. If we are faced with a discontinous function, then it may very well jump to an exceedingly high or low value at any strange point, which would make tracking down extrema a very difficult endeavor.

As stated earlier, the first derivative is the tool we will need to use to find global and local extrema. Let us think about what information we gain from the first derivative. The first derivative provides us with two useful pieces of information about a function - it tells us whether a function is increasing, decreasing, or neither and if it is changing, by how much. The magnitude of the derivative tells us by how much the function is changing - a large magnitude corresponds to a fast rate of change, a small magnitude corresponds to a slow rate of change, and a zero magnitude corresponds to no change. The sign of the derivative tells us how the function is changing: if the derivative is positive, the function is increasing, if it is negative, the function is decreasing, and if it is zero the function is not changing. If the derivative has the same sign over an entire interval, then we can say that the function is either increasing, decreasing, or not changing over that interval. Formally, for a function to be increasing on an interval means that $(x_1 < x_2 \implies f(x_1) < f(x_2))$, and for the function to be decreasing $(x_1 < x_2 \implies f(x_1) > f(x_2))$.

A careful observer may already be able to deduce how we can use the first derivative to find extrema. If we want to identify a local maximum at a point c, we want the function to increase to the left of c, and to decrease to the right of c. This will put c at a peak in the function. If the function is increasing to the left of c, the derivative must be positive to the left of c, and if the function is decreasing to the right of c, the derivative must be negative to the right of c. When we say that the function is increasing to the left of c, we mean there is some interval (b, c) so that the derivative is positive on (b, c), and when we say the function is decreasing to the right of c, we mean there is some interval (c, d) so that the derivative is negative on (c, d).

It is only possible for a derivative to go from positive to negative in one of two ways: either the derivative must cross 0 (which it would if the derivative is a continuous function), or the derivative must be discontinuous, so that it can jump from positive to negative. The point at which the derivative should cross zero (or fail to exist) is c, the local maximum. Similar analysis would indicate that the derivative should be zero (or not exist) at a local minimum, but with the derivative negative to the left, and positive to the right. In conclusion, we can only have a local extremum at a point c if the sign of the derivative changes around c (either from positive to negative - a maximum, or from negative to positive - a minimum). And the only way the derivative can change signs around c is if $f'(c) = 0$ or $f'(c)$ does not exist.

Just as we would expect from the above discussion, if a function f has an extremum at an interior point c of its domain, and $f'(c)$ is defined, then $f'(c) = 0$. However, the converse of this statement is untrue - the fact that $f'(c) = 0$ does not guarantee that c is a local extremum. For instance, we could have a function where the derivative is positive, becomes zero at a point c, and then becomes positive again (x^3 is an example of this, at $x = 0$). Since the function is increasing to the right of c, it is clear that c cannot be a local maximum, and since the function is increasing to the left of c, c also cannot be a local minimum. For any function that isn't a constant, we only have a local extremum at c when the derivative changes signs around c; if the derivative doesn't change signs, then we do not have a local extremum. In conclusion, a zero derivative doesn't guarantee a local extremum, but it is a very good place to try and identify one. We may also find extrema at points where the derivative is undefined (because the derivative could jump and change signs at such a discontinuity), which motivates the following definition.

Critical Point
An interior point of the domain of a function f where f' is either zero or undefined is a **critical point** of f.

Apropos to the previous discussion, the only place in the interior of a function's domain that it can have a local extremum is at a critical point (but a critical point doesn't guarantee there is a local extremum). The only points left to consider are the endpoints of the domain. Thus, if we want to find extrema, we need to look at critical points, and the endpoints of the domain (we have transformed the problem of evaluating the function at an infinite number of points, to evaluating it at a few, select points - much easier). Once we have found the local extrema, it is an easy task to find the global extrema. If a function has a global maximum on a given domain, it will correspond to the local maximum with the largest value. Similarly, a global minimum corresponds to the local minimum with the smallest value.

It is noteworthy that a function may not have a global or local maximum on a given domain. For a continuous function, it turns out that a function defined on a closed interval always achieves its global maximum and minimum. This result is given by the extreme value theorem

Extreme Value Theorem
If f is continuous on the closed interval $[a, b]$, then f attains both a global minimum value m and a global maximum value M in the interval $[a, b]$. That is, there exist $x_1, x_2 \in [a, b]$ so that $f(x_1) = m$ and $f(x_2) = M$, and $m \leq f(x) \leq M$ for all $x \in [a, b]$.

However, for a function defined on an open or half-open interval, the function may or may not achieve its global extrema. Consider the function $f(x) = x$, on the domain $0 \leq x < 1$. The smallest value is $f(0) = 0$, which corresponds to the global minimum. However, there is no global maximum, as no matter how close a point is to $x = 1$, one can choose a point even closer that yields a larger function value. If the function was defined on $0 < x \leq 1$, then it would instead have a global maximum but no global minimum.

Example 1 Find all minima and maxima of $f(x) = xe^{-x}$ on the domain $[0, 4]$.

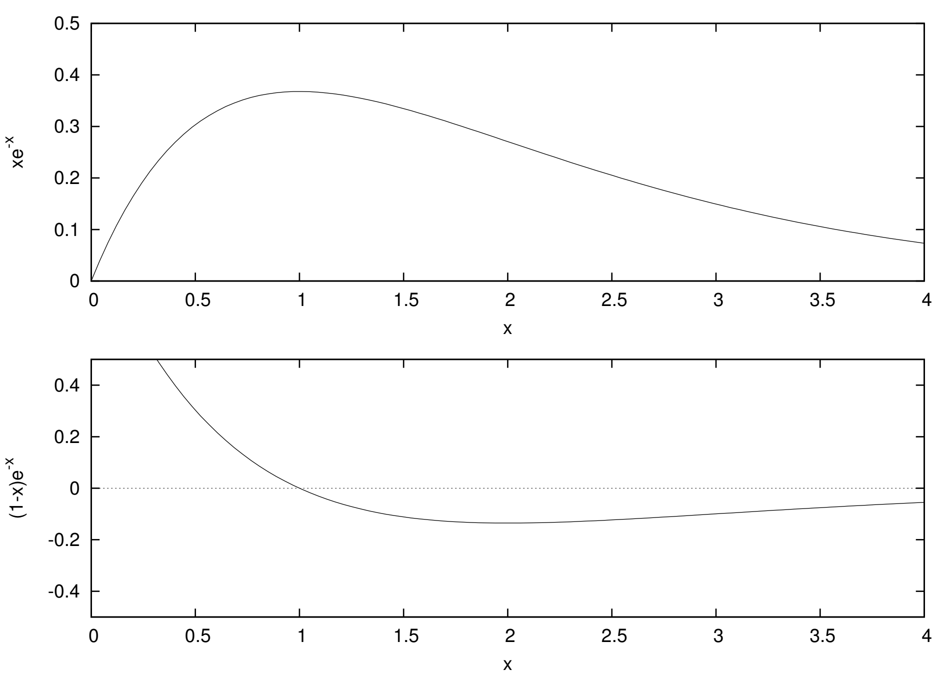
Solution To find local extrema, we need to look at the behavior of the first derivative around critical points, as well as at endpoints. We find

$$f'(x) = 1 \cdot e^{-x} + x \cdot (-e^{-x}) = (1-x)e^{-x}$$

Solving for $f'(x) = 0$ we find $0 = (1-x)e^{-x}$. The right-hand side is 0 only if one of the two terms is zero, and since e^{-x} is always nonnegative, we find that $x = 1$ is the only critical point. According to our previous discussion, we know that the derivative of a function can only change signs around a critical point, so if we partition the interval $[0, 4]$ into subintervals, we know that the sign of the derivative must be constant on $[0, 1)$, as well as on $(1, 4]$. If the sign of the derivatives is different on these two subintervals, we will know that we have found an extremum.

For $x < 1$ we have $1 - x > 0$, and for $x > 1$ we have $1 - x < 0$, so this critical point is a local maximum. On the endpoints we find $f'(0) = 1 > 0$, so the left endpoint is a local minimum (as the function is increasing to the right of it). Similarly, we find $f'(4) = -3e^{-4} \approx -0.055 < 0$, so the right endpoint is also local minimum (as the function is decreasing to the left of it).

To find the global minima and maxima, we need to compare the function value at all critical points, as well as the endpoints of the domain. We find that $f(0) = 0$, $f(1) = e^{-1} \approx 0.367$, and $f(4) = 4e^{-4} \approx 0.073$. Thus, $f(0)$ is both a local and global minimum, and $f(1)$ is both a local and global maximum. It is important to once again note that the global and local maxima are domain specific - that is, the points at which they occur may be different for different domains of the function (which they certainly are in this case).



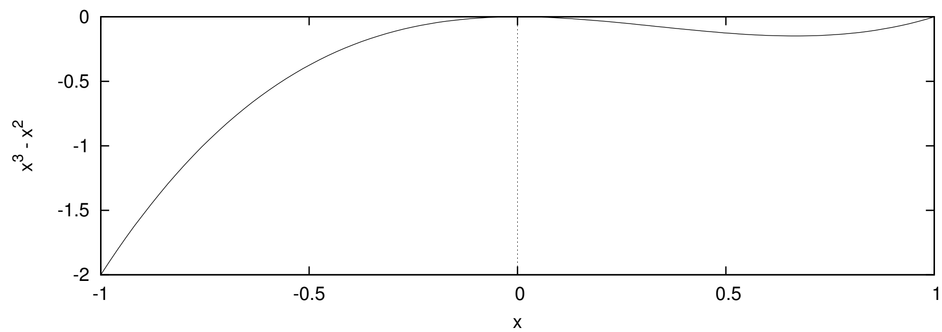
Example 2 Find the global minima and maxima of $f(x) = x^3 - x^2$ on the interval $[-1 : 1]$.

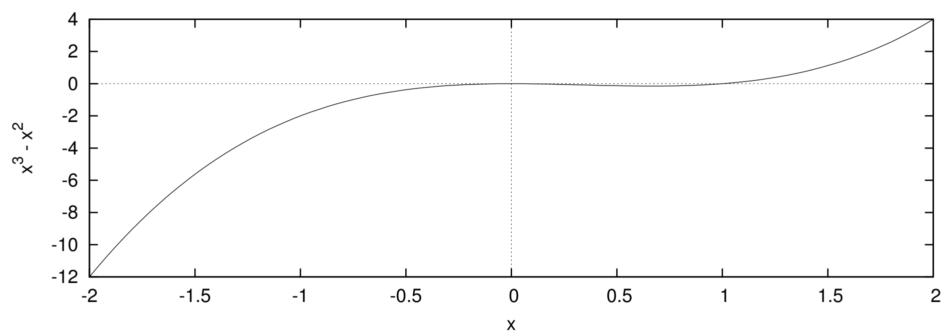
Solution The first step is to identify the critical points. To do so, we compute

$$f'(x) = 3x^2 - 2x$$

And set the derivative equal to 0. Thus $0 = 3x^2 - 2x = x(3x - 2)$, so we have critical points at $x = 0$ and $x = \frac{2}{3}$. Comparing the values of the critical points and endpoints, we find that: $f(-1) = -2$, $f(0) = 0$, $f(\frac{2}{3}) \approx -0.15$, and $f(1) = 0$. Thus, the global minimum is $f(-1)$, while both $f(0)$ and $f(1)$ are global maxima.

Example 3 Find the global minima and maxima of $f(x) = x^3 - x^2$ on the interval $[-2 : 2]$.
Solution We have already found the critical points, so we simply need to compare their values to those of the new endpoints. $f(-2) = -12$ and $f(2) = 4$, so clearly $f(-2)$ is the global minimum, and $f(2)$ is the global maximum. Note that on the new domain there is a single global maximum, rather than two.





Example 4 Find the global maxima and minima of $f(x) = |x|$ over the domain $-2 \leq x \leq 3$.
Solution In order to differentiate this function, it is useful to define it piecewise as follows

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

We can see that for $x > 0$ we have $f'(x) = 1$, and for $x < 0$ we have $f'(x) = -1$, so we don't obtain any critical points there. However, the derivative is not defined at $x = 0$ (because it is a corner), so we have a critical point at $x = 0$ (and it turns out to be a local minimum, because the derivative changes signs from negative to positive as we cross it). The values of the endpoints and this critical point are $f(-2) = 2$, $f(0) = 0$, and $f(3) = 3$. Thus, $x = 0$ is the global minimum, and $x = 3$ is the global maximum.