Limits at Infinity

Two additional topics of interest with limits are limits as $x \to \pm \infty$ and limits where $f(x) \to \pm \infty$. Before we can properly discuss the notion of infinite limits, we will need to begin with a discussion on the concept of infinity. We begin by emphasizing that $\infty$ and $-\infty$ are not a numbers; they are symbols that we use for convenience to represent very specific concepts. Since $\infty$ and $-\infty$ are not numbers, performing arithmetic operations with them is nonsensical. The following argument is an attempt to illustrate this:

**Illustration: $\infty$ as a number is nonsense**

Let us begin by multiplying $\infty$ by 2. Since there is no number larger than $\infty$, it is clear that the result must be $\infty$. Thus,

$$2 \cdot \infty = \infty$$

It follows that

$$2 \cdot \infty - \infty = \infty - \infty = 0$$

because for any number $x$, $x - x = 0$. We can also write that

$$2 \cdot \infty - \infty = (2 - 1) \infty = \infty$$

Finally, we conclude that

$$\infty = 0$$

Thus, the number which is largest in magnitude is equal to the number which is smallest in magnitude.

We were able to draw such a nonsensical conclusion in the above illustration, because we manipulated $\infty$ as a number, which it is not. Now that we’ve considered what $\pm \infty$ are not, the relevant question is what they are. Generally speaking, we use $\infty$ to represent something that is increasing without bound, and use $-\infty$ to represent something that is decreasing without bound. If we say something is increasing without bound, we mean that for any bound $B \in \mathbb{R}$, the something in question eventually becomes larger than the bound $B$. Thus, if something is increasing without bound, it is increasing in a way that it exceeds all real numbers.

In a physical system, we may be interested in the limit as time $t \to \infty$. We can think of this as the state of a physical system after a considerable amount of time has passed (the definition of considerable depending on the nature of the system), because it is not physically possible (or necessarily useful) for us to observe a system for an arbitrarily large amount of time. Unless a system is chaotic, over time it should fall into some sort of steady state behavior, which we can sometimes investigate by looking at limits at infinity. We call the behavior of a function as its input approaches infinity the asymptotic behavior of the function.

A function may exhibit many possible behaviors in the limits as its argument $x \to \pm \infty$. The function can either approach $\pm \infty$, meaning that it increases or decreases without bound, it
can approach a constant value, or it might not approach any value. It should be emphasized that when we say a limit is “equal to infinity,” we are not trying to say that infinity is a real value that a function can have. Rather, we are saying that the output of the function increases without bound. When a function approaches negative infinity, its output decreases without bound. Let’s consider some examples using this intuitive notion of a limit at infinity, before discussing the formal definition.

\[
\lim_{x \to \infty} x = \infty
\]

Since the output is simply the input, it follows that if the input increases without bound, so does the output.

\[
\lim_{x \to \infty} x^n = \infty \quad n > 0
\]

Intuitively, if \( n > 1 \) then \( x^n \) grows faster than \( x \). If \( 0 < n < 1 \) then \( x^n \) is some root of \( x \), but any positive root of \( x \) still grows without bound, so the limit approaches infinity.

\[
\lim_{x \to \infty} x^n = 0 \quad n < 0
\]

We can evaluate the behavior of this limit based on the previous one. If \( x^n \) increases without bound when \( n \) is positive, then the reciprocal of it \( \frac{1}{x^n} \) is 1 divided by a number that increases without bound. As the denominator increases, the fraction decreases, so it follows that the limit approaches 0.

\[
\lim_{x \to \infty} \frac{x}{x + a} = 1 \quad a \in \mathbb{R}
\]

No matter what the value of \( a \) is in the denominator, as \( x \) becomes large enough, the factor of \( a \) will become miniscule in comparison. Then we will essentially be looking at \( \frac{x}{x} \), which is 1.

For any of the above limits, if we multiply the function by \( -1 \), the sign will simply change. Thus, if a function increases without bound, the negative of that function will decrease without bound. Thus

\[
\lim_{x \to \infty} -x^n = -\infty \quad n > 0
\]

Similarly, if we look in the limit as \( x \to -\infty \), the only thing we might have to worry about is a negative sign - otherwise the analysis is the same. If a function oscillates with time, then it will not settle down to a single value, or increase/decrease without bound. Thus

\[
\lim_{x \to \infty} \cos(x) \text{ does not exist}
\]

If we modulate the cosine function with a decreasing amplitude, then we will have a function that approaches 0.

\[
\lim_{x \to \infty} x^{-2} \cos(x) = 0
\]

As previously stated, the precise definition of \( \infty \) depends on the context. Formally, limits as \( x \to \pm \infty \) are defined as follows:
**Definition: Limits as \( x \to \infty \) or \( x \to -\infty \)**

We write \( \lim_{x \to \infty} f(x) = L \)

if for every number \( \epsilon > 0 \) there exists a corresponding number \( M \) such that for all \( x \) with \( x > M \) we have

\[ |f(x) - L| < \epsilon \]

Similarly, we write

\( \lim_{x \to -\infty} f(x) = L \)

if for every number \( \epsilon > 0 \) there exists a corresponding number \( N \) such that for all \( x \) with \( x < N \) we have

\[ |f(x) - L| < \epsilon \]

The method of proving a limit exists at \( \pm \infty \) is very similar to proving that a limit exists at a point. The only difference is that rather than creating a \( \delta \) interval around the point of interest to restrict the function values within the error tolerance of the limit value, we need to find a value \( M \) or \( N \) so that if the input values \( x \) are large/small enough the function values are within the error tolerance of the limit. Just as the definition of limits at \( \pm \infty \) and a limit at a point are extremely similar, all of the properties we stated previously for combining limits also hold for limits at \( \pm \infty \).

**Example 1** Prove that

\[ \lim_{x \to \infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x} = 0 \]

**Solution** We will begin with the limit as \( x \to \infty \). Consider \( \epsilon > 0 \), arbitrary. We need to find \( M \) such that for all \( x \) with \( x > M \) we have

\[ |f(x) - L| = |\frac{1}{x} - 0| = \frac{1}{x} < \epsilon \]

Note that above we were able to drop the absolute value signs because if \( x \to \infty \) then clearly \( x > 0 \). Multiplying both sides of the equation by \( x \) and dividing by \( \epsilon \) we find

\[ x > \frac{1}{\epsilon} \]

Thus, as long as \( x > 1/\epsilon \) we will have

\[ |\frac{1}{x} - 0| < \epsilon \]

Setting \( M = 1/\epsilon \) (or any value larger than \( 1/\epsilon \)) will be sufficient to guarantee this. The conclusion follows.

Proving the limit as \( x \to -\infty \) is completely analogous, which can be shown by setting \( N \leq -1/\epsilon \).
If we are faced with a rational function, that is, a function which is one polynomial divided by another, we can divide by the highest power in the denominator in order to evaluate the limit. Consider the following examples.

**Example 2** Find the following limit

\[
\lim_{x \to \infty} \frac{2x^2 - 1}{5x^2 - x}
\]

Since the highest power of \(x\) in the denominator is 2 (ie, \(x^2\)) we want to multiply our rational function by a convenient choice of \(1/x^2\), in order to simplify the analysis (our choice will be \(1/(5x^2)\)). The effect of this multiplication will be to remove all terms in the denominator approaching \(\pm \infty\), in order to reduce the limit into something we can analyze (because considering something of the form \(\infty/\infty\) is meaningless). Doing so,

\[
\lim_{x \to \infty} \frac{2x^2 - 1}{5x^2 - x} = \lim_{x \to \infty} \frac{2 - 1/x^2}{5 - 1/x} = \frac{2 - 0}{5 - 0} = \frac{2}{5}
\]

Thus, even though the polynomial in the numerator and denominator both approach \(\infty\) as \(x \to \infty\), in looking at their quotient, it is possible to find a finite value.

**Example 3** Find the following limit

\[
\lim_{x \to \infty} \frac{2x - 1}{5x^2 - x}
\]

We will once again perform the same trick to find that

\[
\lim_{x \to \infty} \frac{2x - 1}{5x^2 - x} = \lim_{x \to \infty} \frac{2 - 1/x}{5 - 1/x^2} = \frac{0 - 0}{5 - 0} = 0
\]

In this case the denominator approaches \(\infty\) more quickly than the numerator, so the limit of the quotient is 0.

**Example 4** Find the following limit

\[
\lim_{x \to -\infty} \frac{2x^4 - x^3}{5x^3 - x}
\]

Yet again we will perform the same trick, finding that

\[
\lim_{x \to -\infty} \frac{2x^4 - x^3}{5x^3 - x} = \lim_{x \to -\infty} \frac{2 - 1/x^3}{5 - 1/x^2} = \frac{-\infty}{-\infty} = -\infty
\]

In this example the numerator approaches \(\infty\) while the denominator approaches \(-\infty\). Since the magnitude of the numerator is growing faster than that of the denominator, the magnitude of the limit is \(\infty\). We pick up a negative sign because the denominator is negative.
If a function has a finite limit as $x \to \pm \infty$ we say that the horizontal line with the same value as the limit is a horizontal asymptote of the function, and that the function approaches that value asymptotically. Since
\[
\lim_{x \to \infty} \frac{2x^2 - 1}{5x^2 - x} = \frac{2}{5}
\]
we say that $2/5$ is a horizontal asymptote of this function. Similarly, the $x$-axis is a horizontal asymptote of $1/x$. Note that if a function does not approach finite values as $x \to \pm \infty$, then it does not have any horizontal asymptotes (but it may have vertical asymptotes, which we will consider in the next section).

We will conclude with a nice little trick for evaluating limits at infinity - substitution.

**Example 5** Find $\lim_{x \to \infty} \sin(\frac{1}{x})$

**Solution** Here we will use the fact that as $x \to \infty$ we have $1/x \to 0$, and introduce the variable $t = 1/x$. Thus,
\[
\lim_{x \to \infty} \sin\left(\frac{1}{x}\right) = \lim_{t \to 0^+} \sin(t) = 0
\]
The reason we need to write this as a one-sided limit is because $x \to \infty$ from only one side.