Finding Volumes Using Slabs

While we have been successful in using integration to find the area of many objects, the notion of integration can also be extended to calculate volumes, which is our current goal. When we calculate the area of a region, we simply divide the region into a number of small pieces, each of which we can calculate its area. By summing all of these areas, we find the total area. Similarly, we will want to divide a three-dimensional region into a number of small volumes, and sum the volumes of each of the smaller pieces. The result will give us the total volume of the region. It's worth noting that it does not matter how we cut the solid into smaller pieces; there are multiple methods for doing so, and the optimal method depends on the situation.

For now we are interested in approximating a volume using cylindrical slabs. In order to find the volume of a cylinder, we simply calculate the surface area of the base, and multiply it by the height. The reason why this formula works is because the cross-section of the cylinder doesn't change. No matter where we think about slicing (perpendicular to the height), the two-dimensional figure will be identical to that of the base. What if we have a more complicated object, so that the cross-sections vary as we move along its height? For instance, we could think of a perfect loaf of bread as our cylindrical object, and a lopsided loaf as our more complicated object. When we cut out slices of bread from the perfect loaf, they all look the same, but the slices from the lopsided loaf vary depending on where we cut from. However, if we cut the slices thin enough, and think of each slice as a new object in its own right, the cross-sections of the slice vary very little. That is, each of the individual slices is approximately like a cylinder, and the approximation improves the thinner we make the slices. Each slice will have a volume given by

\[ A(x)\,dx \]

where \( A(x) \) is the area of the base, and \( dx \) represents the height of the very thin slice. If we add the volume of all of these slices together, we will get the total volume, so if our lopsided loaf extends from \( a \) to \( b \), its volume will be given by

\[ V = \int_a^b A(x)\,dx \]

**Example 1** A pyramid that is 3m in height has a square base with 3m on each side. The cross-section of the pyramid perpendicular to the altitude \( x \) m down from the vertex is a square \( x \) m on a side. Find the volume of the pyramid.

**Solution** In this case it is helpful to define a coordinate system so that the apex of the pyramid is at the origin, and the altitude of the pyramid is along the \( x \)-axis. In this way, the pyramid extends from 0 to 3 along the \( x \)-axis. A given cross-section is \( x \) by \( x \) m², so the volume of a given slab is \( x^2\,dx \) and the total volume is given by

\[ V = \int_0^3 x^2\,dx = \frac{x^3}{3}\bigg|_0^3 = 9 \, \text{m}^3 \]

**Example 2** Find the volume of a right circular cone with height \( h \) and radius \( r \).
Solution Once again we will want to place the tip of the cone at the origin, and let its height extend along the $x$-axis, so the cone will begin at $x = 0$ and end at $x = h$. In order to find the area of a cross-section, we will need to use similar triangles. Since we know each cross-section is circular, we simply need to find the radius of a general cross-section. Moving out a distance $x$ from the origin, let $y$ denote the height of the triangle formed by the $x$-axis, the slant of the cone, and a line drawn at a right angle to the $x$-axis. This triangle is similar to the triangle given by the slant, the height $h$, and the radius $r$. Thus, we find that

$$\frac{y}{x} = \frac{r}{h}$$

or

$$y = \frac{r}{h}x$$

where $y$ will give us the radius of a general cross-section. Finding the area then

$$\pi y^2 = \pi \frac{r^2}{h^2}x^2$$

Finally, we are ready to integrate, and find that

$$V = \int_0^h \pi \frac{r^2}{h^2}x^2 \, dx = \pi \frac{r^2}{h^2} \frac{x^3}{3} \bigg|_0^h = \pi r^2 h \frac{h^3}{3}$$

which is the familiar formula for the volume of a right-circular cone.

In light of the above example, we can think of generating a solid by considering a function $f$ and rotating it about a line, for instance, the $x$-axis. The result will be a three-dimensional figure, with each of its cross-sections given by a circle. The function $f$ and the line chosen will determine the radius of each cross-section, and then volume of a single slab will be given by

$$\pi r^2 \, dx$$

where $r$ is a function of $x$ or

$$\pi r^2 \, dy$$

for $r$ a function of $y$.

Example 3 The region between the curve $1 + \sin(x)$, $0 \leq x \leq 4\pi$ and the $x$-axis is revolved around the $x$-axis to generate a solid. Find its volume.

Solution In this case the area of a cross section is given by

$$A(x) = \pi (1 + \sin(x))^2 = \pi (1 + 2\sin(x) + \sin^2(x)) = \pi \left(\frac{3}{2} + 2\sin(x) - \frac{\cos(2x)}{2}\right)$$

noting that

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
and that we need to integrate from 0 to $4\pi$, we find

$$V = \pi \int_0^{4\pi} \left( \frac{3}{2} + 2 \sin(x) - \frac{\cos(2x)}{2} \right) dx = \pi \left[ \frac{3}{2}x - 2 \cos(x) - \frac{\sin(2x)}{4} \right]_0^{4\pi} = \pi (6\pi - 2 - 0 - (0 - 2 - 0)) = 6\pi^2$$

**Example 4** A sphere can be generated by rotating the circle

$$x^2 + y^2 = a^2$$

about the $x$-axis. Find the volume of the sphere.

**Solution** The radius of a typical cross-section is given by $y$, so

$$A(x) = \pi y^2 = \pi (a^2 - x^2)$$

In this case the sphere extends from $-a$ to $a$ on the $x$-axis, so the volume is given by

$$V = \pi \int_{-a}^{a} (a^2 - x^2) dx = \pi (a^2 x - \frac{x^3}{3}) \bigg|_{-a}^{a} = \frac{4}{3} \pi a^3$$

**Example 5** Find the volume generated by revolving the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$ about the line $y = 1$.

**Solution** Even though the revolution is not about the $x$-axis, we still simply find cross-sections in the same way as before, in order to find the area. First, note that along the $x$-axis we will need to integrate from 1 to 4. Next, see that a typical cross-section is given by

$$A(x) = \pi (\sqrt{x} - 1)^2 = \pi (x - 2\sqrt{x} + 1)$$

so

$$V = \int_{1}^{4} \pi (x - 2\sqrt{x} + 1) dx = \pi \left( \frac{x^2}{2} - 2 \cdot \frac{2}{3}x^{3/2} + x \right) \bigg|_{1}^{4} = \frac{7\pi}{6}$$

**Example 6** Find the volume of the solid generated by revolving the region between the line $x = 2$ and the curve $x = 2/y + 2$, $1 \leq y \leq 4$, about the line $x = 2$.

**Solution** The only difference between this situation and the previous ones is now we will have cross-sections which are given as functions of $y$, and we will sum them vertically, rather than horizontally as we have been doing previously. The area of a cross-section is given by

$$A(y) = \pi (2/y + 2 - 2)^2 = \frac{4\pi}{y^2}$$

Here our limits of integration are 1 to 4, so we find

$$V = 4\pi \int_{1}^{4} \frac{dy}{y^2} = -\frac{4\pi}{y} \bigg|_{1}^{4} = -\pi + 4\pi = 3\pi$$

What now, if we generated a solid using the above curve, but rather than rotating about $x = 2$, we rotate about the $y$-axis? The result will be a volume with a hollow inside. In order to find the area of a cross-section of this new solid, we simply find the area as if there
were no hole, and then subtract the area from the inside. Thus, if the outside has a radius of \( R \), and the inside has a radius of \( r \), the cross-sectional area will be given by

\[
A(x) = \pi (R^2 - r^2)
\]

**Example 7** Find the volume of the solid generated by revolving the region between the line \( x = 2 \) and the curve \( x = 2/y + 2 \), \( 1 \leq y \leq 4 \), about the \( y \)-axis.

**Solution** The area of a cross-section is given by

\[
A(y) = \pi (2/y + 2)^2 - \pi 2^2 = \pi \left( \frac{4}{y^2} + \frac{4}{y} + 4 - 4 \right) = 4\pi \left( \frac{1}{y^2} + \frac{1}{y} \right)
\]

Here our limits of integration are 1 to 4, so we find

\[
V = 4\pi \int_{1}^{4} \left( \frac{1}{y^2} + \frac{1}{y} \right) dy = 4\pi \left[ -\frac{1}{y} + \ln(|y|) \right]_{1}^{4} = 4\pi \left( -\frac{1}{4} + \ln(4) + 1 - \ln(1) \right) = 3\pi + 4\pi \ln(4) \approx 26.8455
\]

In a more general setting, the hollowed out inside of the function need not been a simple cylinder. Consider the following example.

**Example 8** The region bounded by the curve \( y = x^2 + 1 \) and \( y = -x + 3 \) is revolved about the \( x \)-axis to generate a solid. Find the volume of the solid.

**Solution** First we need to find the intersection of these two curves to determine the region we are revolving.

\[
\begin{align*}
x^2 + 1 &= -x + 3 \\
x^2 + x - 2 &= 0 \\
(x + 2)(x - 1) &= 0
\end{align*}
\]

so our points of intersection are \((-2, 5)\) and \((1, 2)\). Choosing a sample point of \( x = 0 \), we see that \(-x + 3\) is the higher of these two curves, so it will define the outer radius of our cross-section. We find that

\[
A(x) = \pi ((-x + 3)^2 - (x^2 + 1)^2) = \pi (8 - 6x - x^2 - x^4)
\]

It follows then that

\[
V = \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4) dx = \pi \left[ 8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^{1} = \frac{117\pi}{5}
\]