Real Analysis, Abbott

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1 A preliminary proof

Theorem 1 (Equality of real numbers.)
Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a-b| < \epsilon$.

Proof. ($\Rightarrow$) Assume $a = b$. Then $|a - b| = 0 < \epsilon$, as desired. ($\Leftarrow$) Assume $|a - b| < \epsilon$ for all $\epsilon > 0$. Now assume for contradiction that $|a - b| = \epsilon_0$. By assumption we must have $|a - b| < \epsilon_0$ also, a contradiction. Hence $|a - b| < \epsilon$ for all $\epsilon > 0$, and we are done.

2 Upper and lower bounds

Proposition 2 (Axiom of completeness.)
Every non-empty set of real numbers that is bounded above has a least upper bound.

Definition 3 (Upper bound, bounded above; lower bound, bounded below.)
A set $A \subseteq \mathbb{R}$ is *bounded above* (resp. *bounded below*) if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ (resp. $a \geq b$) for all $a \in A$. The number b is called an *upper bound* (resp. a *lower bound*) for A.

Definition 4 (Least upper bound, supremum; greatest lower bound, infimum.)
A real number s is the *least upper bound* (resp. *greatest lower bound*) or *supremum* (resp. *infimum*) of a set A if s is an upper bound (resp. lower bound) and $s \leq s'$ (resp. $s \geq s'$) for any s' an upper bound (resp. lower bound) of A. We denote s by $s = \sup A$ (resp. $s = \inf A$).

Definition 5 (Maximum, minimum.)
A real number s is the *maximum* (resp. *minimum*) of a set A if $s \in A$ and $s \geq s'$ (resp. $s \leq s'$) for all $s' \in A$.

Theorem 6 (Maximum and supremum; minimum and infimum.)
If set A has a maximum (resp. minimum) s, then its supremum (resp. infimum) is also s.

Proof. By definition of maximum (resp. minimum), s is an upper bound (resp. a lower bound). Because $s \in A$, $s < s'$ (resp. $s > s'$) for any s' an upper bound (resp. a lower bound) of A. Hence $s = \sup A$ (resp. $s = \inf A$). Note that a maximum (resp. minimum) need not always exist.

Theorem 7 (Alternative definition of supremum.)
A real number s that is an upper bound of the set A is also its supremum if and only if there exists an $a \in A$ such that $s - \epsilon < a$ for all $\epsilon > 0$.

Proof. ($\Rightarrow$) Assume $s = \sup A$. Now assume for contradiction that $s - \epsilon \geq a$ for all $a \in A$. Then $s - \epsilon$ is an upper bound for A smaller than $s = \sup A$, a contradiction. ($\Leftarrow$) Assume that there exists an $a \in A$ such that $s - \epsilon < a$ for all $\epsilon > 0$. Then there can be no upper bound smaller than s, and therefore $s = \sup A$.

A similar statement and proof can be made for the infimum, but the parentheses are getting tiring.

Theorem 8 ("Axiom of completeness" for lower bounds.)
Let A be bounded below, and define B as the set of all lower bounds b of the set A. Then $\sup B = \inf A$.

Proof. By the axiom of completeness, $\sup B$ exists. Let $b' = \sup B$ and $a' = \inf A$. Since b' is a lower bound of A, $a' \geq b'$; since b' is the greatest lower bound of A, $b' \geq a'$. Hence $a' = b'$, and we are done.

This shows that we need not postulate that a greatest lower bound exists for sets bounded below.

Theorem 9 (Partial linearity of supremum.)
Given sets A and B, each bounded above, and a real constant $c > 0$,

1. $\sup(A + c) = c + \sup A$,
2. $\sup(cA) = c \sup A$,
3. $\sup(A + B) = \sup A + \sup B$.

Proof.

1. Let $a' = \sup A$, and a'' any upper bound of A. Then $a'' \geq a' \geq a$ for all $a \in A$, and hence $a'' + c \geq a' + c \geq a + c$ for all $(a + c) \in (c + A)$. Because $a'' + c$ is an arbitrary upper bound of $c + A$, $c + a' = \sup(c + A)$.
2. Let $a' = \sup A$, and a'' any upper bound of A. Then $a'' \geq a' \geq a$ for all $a \in A$, and hence $ca'' \geq ca' \geq ca$ for all $ca \in cA$. Because ca'' is an arbitrary upper bound of cA, $ca' = \sup(cA)$.
3. Let $a' = \sup A$, $b' = \sup B$, and a'', b'' any upper bound of A and B respectively. Then $a'' \geq a' \geq a$ for all $a \in A$ and $b'' \geq b' \geq b$ for all $b \in B$, and hence $a'' + b'' \geq a' + b' \geq a + b$ for all $(a + b) \in (A + B)$. Because $a'' + b''$ is an arbitrary upper bound of $A + B$, $a' + b' = \sup(A + B)$.

What happens if $c < 0$?

Some alternative proofs

Theorem 10 (Theorem 2.3.4.i)
If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

Proof. We will prove the contrapositive of the above statement: if $a < 0$, then there exists $a_n < 0$ for some $n \in \mathbb{N}$. Let $a = -\epsilon_0$. Then, for all $\epsilon > 0$, for some $n \geq N$, $|a_n + \epsilon_0| = |a_n - a| < \epsilon$. Set $\epsilon = \epsilon_0$, so that

$$|a_n + \epsilon_0| < \epsilon_0 \quad \Longleftrightarrow \quad -\epsilon_0 < a_n + \epsilon_0 < \epsilon_0 \quad \Longleftrightarrow \quad -2\epsilon_0 < a_n < 0,$$

as desired.

Theorem 11 (Exercise 2.5.6)
Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Proof. Consider the subsequence (a_{n_k}) defined by

$$a_{n_k} \text{ is the } k^{\text{th}} \text{ term in } (a_n) \text{ such that } a_{n_k} \geq \sup S.$$

Assume for contradiction that this sequence does not converge to $\sup S$. Then

$$\text{there exists } \epsilon_0 > 0 \text{ such that for all } K, \text{ there exists } k > K \text{ such that } a_{n_k} - \sup S > \epsilon_0.$$

There is an infinite number of such k, for one can always take $K = 1 + \max\{k_n\}$, where $\{k_n\}$ is any finite subset of k's. But then $x = \sup S + \epsilon_0 \in S$, which contradicts $x \leq \sup S$. Hence (a_{n_k}) converges to $\sup S$.

Theorem 12 (Theorem 3.3.4)
A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. ($\Rightarrow$) Let a compact set K be arbitrary. By definition, every sequence in K has a subsequence that converges to a limit in K. Consider first all convergent sequences in K. All subsequences of these convergent sequences must converge to the same limit, which is in K by definition. Therefore all convergent sequences in K have their limit in K, and hence K contains all its limit points and is closed. Further, by contraposition, K must be bounded. Let K' be an unbounded set. Then it contains a monotone unbounded sequence, each of whose subsequences is unbounded. Hence K' is not compact, so K must be bounded. ($\Leftarrow$) Let a closed and bounded set L be arbitrary. Then by the Bolzano-Weierstrass theorem every sequence in L possesses a convergent subsequence whose limit is in L because L is closed. Hence L is compact.

Theorem 13 (Theorem 3.4.7)
A set $E \subseteq \mathbb{R}$ is connected if and only if $c \in E$ for all c such that $a < c < b$, where $a, b \in E$.

Proof. [Not new, just restated in one direction.] ($\Rightarrow$) Let E be connected, and consider $A = (-\infty, c) \cap E$ and $B = (c, \infty) \cap E$. Then A and B are nonempty and separated. If $c \notin E$, then $E = A \cup B$, but this implies that E is disconnected, a contradiction. Hence $c \in E$.

Theorem 14 (Unstated theorem?)
Any open subset of $\mathbb{R}$ is a finite or countable union of disjoint open intervals.

Proof. A finite union is a countable union, so we need only prove countability. We know that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, so our goal is to construct a set of disjoint open intervals based on the elements of $\mathbb{Q}$. Such a set is countable by construction.

Let $A \subseteq \mathbb{R}$ be open and arbitrary. Every element $a \in A$ is either rational or irrational. If a is rational, then define the interval

$$I_a = \bigcup_{\substack{I \text{ an open interval} \\ a \in I \subseteq A}} I.$$

If a is irrational, then it exists within an ϵ-neighborhood of a rational number a', and there must exist such an ϵ-neighborhood in A because A is open. Then $a \in I_{a'}$. Since all elements $a \in A$ are in an interval I_q for all $q \in A \cap \mathbb{Q}$, then

$$A \subseteq \bigcup_{q \in A \cap \mathbb{Q}} I_q \quad \text{and, by construction,} \quad \bigcup_{q \in A \cap \mathbb{Q}} I_q \subseteq A, \quad \text{so that} \quad A = \bigcup_{q \in A \cap \mathbb{Q}} I_q.$$

It remains to show that the intervals I_q are disjoint, or equivalently that if $x \in I_c \cap I_d$, then $I_c = I_d$. Since $x \in I_c$ and $x \in I_d$, by construction $I_c \subseteq I_d$ and $I_d \subseteq I_c$, so $I_c = I_d$.

Theorem 15 (Theorem 4.2.3)
$\lim_{x \to c} f(x) = L$ if and only if, for all sequences $(x_n) \subseteq A$ such that $x_n \neq c$ and $(x_n) \to c$, $f(x_n) \to L$.

Proof. The former clause is equivalent to the statement that

$$\text{for all } \epsilon > 0, \text{ there exists } \delta > 0 \text{ with } 0 < |x - c| < \delta \text{ such that } |f(x) - L| < \epsilon,$$

and the latter to the statement that

$$\text{for all } \epsilon > 0, \text{ there exists } N \text{ such that } |f(x_n) - L| < \epsilon \text{ for all } n \geq N.$$

($\Rightarrow$) Assume that $\lim_{x \to c} f(x) = L$ and that $(x_n) \to c$, $x_n \neq c$. We need to find an N such that the conditions of the second clause are fulfilled. Let ϵ be arbitrary. Then there exists a corresponding δ such that $0 < |x - c| < \delta$. Choose N to be the minimum n for which $|x_n - c| < \delta$ for all $n \geq N$. Such an N must exist, since $(x_n) \to c$, and by hypothesis $|f(x_n) - L| < \epsilon$ for all $n \geq N$. ($\Leftarrow$) Assume the second clause. We need to find a δ such that the conditions of the first clause are fulfilled. Let ϵ be arbitrary, and choose $\delta > x_N - c$, where N is as stated in the second clause; by hypothesis, the conditions of the first clause are fulfilled.

Theorem 16 (Exercise 4.3.7)
Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous, and let $K = \{x : h(x) = 0\}$. Then K is a closed set.

Proof. If no such x exists, then K is the null set, which is closed. We will thus assume that K is non-empty and show that K^c is open. Let $x \notin K$ be arbitrary. Then by continuity of h any ϵ-neighborhood about $h(x)$ with $0 < \epsilon < |h(x)|$ possesses a δ-neighborhood about x such that $V_\delta(x) \subseteq K^c$, which defines an open set.

Theorem 17 (Exercise 4.3.8)
A function continuous on $\mathbb{R}$ and equal to 0 at every rational point must be identically 0 on $\mathbb{R}$.

Proof. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, any irrational number is arbitrarily close to a rational number, and by continuity must therefore take on the same value. Because $\mathbb{R}$ is the union of $\mathbb{Q}$ and $\mathbb{I}$, the function must be identically 0 on $\mathbb{R}$.