1 The real numbers

Salient sets.

- $\mathbb{N} = \{1, 2, 3, \cdots\}$, the natural numbers
- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, the integers
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$, the rational numbers
- $\mathbb{I} = \{x : x \notin \mathbb{Q}\}$, the irrational numbers
- $\mathbb{R} = \{x : x \in \mathbb{Q} \cup \mathbb{I}\}$, the real numbers

With regards to these sets, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ and $\mathbb{I} \subseteq \mathbb{R}$, as can be seen by definition. $\mathbb{Q}$ and $\mathbb{I}$ are disjoint, having no elements in common, and their union is $\mathbb{R}$.

Methods of proof.

- induction: show true for base case, assume true for n, show true for $n+1$—works only for finite n; that is, we can't take $n = \infty$, becuase $\infty \notin \mathbb{N}$
- contradiction: assume the negation of the statement to be proven, and hence obtain a contradiction
- contrapositive: instead of proving $A \implies B$, prove $\neg B \implies \neg A$
- construction: prove that something exists by example

Bounds. The infimum is the greatest lower bound; the supremum is the least upper bound. All sets have infima and suprema, and finite sets necessarily have minima and maxima. Infinite sets may or may not have minima or maxima. Where minima or maxima exist, they are respectively infima and suprema. Minima and maxima are necessarily members of the set which they describe; not so infima and suprema.

Axiom of Completeness. Every nonempty set of real numbers that is bounded above has a supremum. This is one of a few equivalent statements for formally stating that $\mathbb{R}$ doesn't contain the gaps in $\mathbb{Q}$, although this form of the statement is not particularly obvious. In fact, we may even treat this axiom as a definition of the set of real numbers. It also follows from this axiom that every nonempty set of real numbers bounded below has an infimum: consider the set of upper bounds of a set bounded above.

Nested Interval Property. For each $n \in \mathbb{N}$, assume that we have an interval $I_n = [a_n, b_n]$ with the property that $I_n \supseteq I_{n+1}$. Then the nested sequence of closed intervals

$$I_1 \cap I_2 \cap I_3 \cap \cdots = \bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

The proof follows from noting that $s = \sup\{a_n : n \in \mathbb{N}\}$ lies within all intervals I_n, using the properties of the supremum and the Axiom of Completeness. The Nested Interval Property is equivalent to the Axiom of Completeness: indeed, we could have assumed the Nested Interval Property and hence prove the Axiom of Completeness. To do this, consider a set S with a an upper bound for S. Let $I_n = [s, x_n]$ and define

$$I_{n+1} = \begin{cases} [s, x_{n+1}] & x_{n+1} \text{ an upper bound,} \\ [x_{n+1}, a] & x_{n+1} \text{ not an upper bound,} \end{cases} \quad \text{where } x_{n+1} = \frac{s + x_n}{2} \text{ and } x_1 = a,$$

and consider the resulting $\{I_n\}$.

Density. (Density of $\mathbb{Q}$ in $\mathbb{R}$.) There exists a rational number r between any two real numbers a, b, $a < b$. By construction, let $r = m/n$, with $m, n \in \mathbb{N}$. Choose n such that $0 < 1/n < b - a$, and choose m such that $(m-1)/n < a < m/n$. This construction is motivated by choosing a gap $(1/n)$ small enough that m times this gap lies between a and b. (Density of $\mathbb{I}$ in $\mathbb{R}$.) From the above, $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\mathbb{Q} + \sqrt{2}$ is dense in $\mathbb{R} + \sqrt{2} = \mathbb{R}$. But $\mathbb{Q} + \sqrt{2}$ is a subset of $\mathbb{I}$, so $\mathbb{I}$ must be dense in $\mathbb{R}$. Specializing from the above, there always exists a rational between two rationals, and an irrational between two irrationals.

Cardinality. Two sets A and B have the same cardinality if there exists a bijection between A and B. If such a bijection exists, we write $A \sim B$. A set C is countable if $\mathbb{N} \sim C$. Otherwise C is uncountable. In particular, $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable, and $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$ is therefore uncountable. Proofs of countability are often done by construction of such a bijection, and of uncountability often by contradiction.