The Carnot Cycle

We will investigate some properties of the Carnot cycle and explore the subtleties of Carnot's theorem.

1. Recall that the Carnot cycle consists of the following four reversible steps, in order:

   (a) isothermal expansion at T_H

   (b) isentropic expansion at $S_{\max}$

   (c) isothermal compression at T_C

   (d) isentropic compression at $S_{\min}$

   T_H and T_C represent the temperatures of the hot and cold thermal reservoirs; $S_{\min}$ and $S_{\max}$ represent the minimum and maximum entropy reached by the working fluid in the Carnot cycle. **Draw** the Carnot cycle on a TS plot and **label** each step of the cycle, as well as T_H, T_C, $S_{\min}$, and $S_{\max}$.

2. On the same plot, **sketch** the cycle that results when steps (a) and (c) are respectively replaced with irreversible isothermal expansion and compression, assuming that the resulting cycle operates between the same two entropies $S_{\min}$ and $S_{\max}$.

We will prove geometrically that the Carnot cycle is the most efficient thermodynamic cycle working between the range of temperatures bounded by T_H and T_C. To this end, consider an arbitrary reversible thermodynamic cycle C with working fluid having maximum and minimum temperatures T_H and T_C, and maximum and minimum entropies $S_{\max}$ and $S_{\min}$. It is sufficient to consider reversible cycles, for irreversible cycles must have lower efficiencies by Clausius's inequality.

3. Define two areas A and B on the corresponding TS plot for C. A is the area enclosed by C, whereas B is the area bounded above by the bottom of C, bounded below by the S-axis, and bounded to left and right by the lines $S = S_{\min}$ and $S = S_{\max}$. **Express** the thermodynamic efficiency of C in terms of the areas A and B.

4. **Argue** that there exists a Carnot cycle C' with working fluid having the same maximum and minimum temperatures and entropies as C. Hence **deduce**, using the result from (3.), that C' possesses an equal or greater thermodynamic efficiency than does C, with equality holding only for C a Carnot cycle. Because C is arbitrary, this implies that *no thermodynamic cycle is more efficient than a Carnot cycle.*

5. Our results are summarized by *Carnot's theorem*:

   (a) any reversible engine operating between two thermal reservoirs at temperatures T_H and T_C has the same efficiency, and

   (b) any irreversible engine operating between two thermal reservoirs at temperatures T_H and T_C has a lower efficiency than that of a reversible engine.

   The latter point is easy to comprehend, but the former is harder. If we consider a Carnot cycle and, say, an arbitrary reversible non-Carnot cycle operating between the same temperature range, we have shown in (4.) that the Carnot cycle will be more efficient than the arbitrary cycle. But this seems to contradict (5.a)!

   Resolve this contradiction by showing that the only reversible cycle operating between two thermal reservoirs is the Carnot cycle, or conversely that all other reversible cycles must operate between more than two thermal reservoirs. (Hint: **Consider** carefully the steps in the Carnot cycle.)

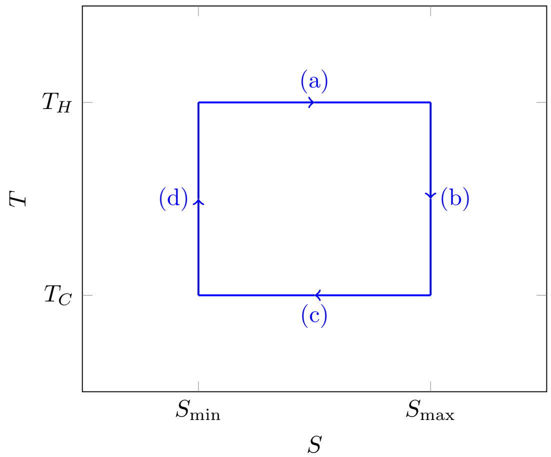
Solutions

1. Please refer to figure (a) in the following page.
2. Please refer to figure (b) in the following page. Figure (b) looks exactly like figure (a), but the updated steps (a) and (c) have been drawn in dashed lines to emphasize their irreversibility. That the figures should be the same can be understood in the following manner: steps (b) and (d) remain the same, as do $S_{\min}$ and $S_{\max}$, so those steps must not change. Because irreversible isothermal expansion and compression still occur at the same temperatures T_H and T_C, those steps must not change either.

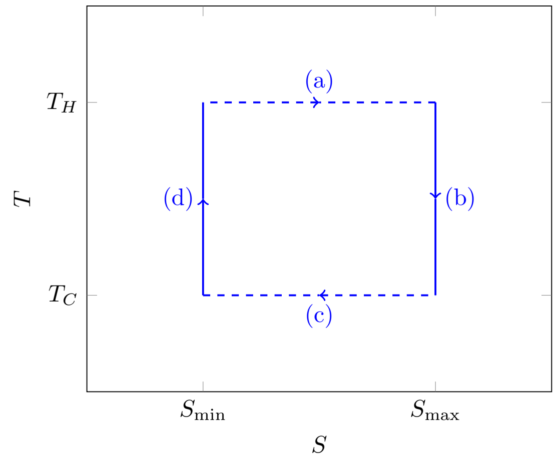
   What *has* changed? More work was required to effect compression and less gained from expansion: this work was dissipated as heat to the environment, leading to a net increase in entropy.
3. The thermodynamic efficiency is simply the ratio of work output to heat input, $\eta = A/(A + B)$.
4. We will construct a Carnot cycle with the given properties. Let the hot reservoir have temperature T_H and the cold reservoir temperature T_C. We shall expand the fluid until it has entropy $S_{\max}$, and compress it until it has entropy $S_{\min}$. This is a Carnot cycle with the same parameters as C.

   Now consider figure (c) in the following page, representing the arbitrary cycle C. The Carnot cycle C' will have $A' \geq A$ and $B' \leq B$, as is obvious geometrically. Referring to the formula from (3.), each of these changes will increase (or, given equality, not change) the thermodynamic efficiency of C' relative to C, so the stated result is proven.
5. The only reversible processes that require less than two thermal reservoirs are isothermal (one reservoir) and isentropic (no reservoirs). In the former case only one reservoir is needed to maintain the temperature of the fluid; in the latter case no heat transfer occurs, and no reservoir is employed. All other processes involve heat transfer and a change in temperature for the fluid, and each infinitesimal change in temperature of the fluid requires a different reservoir at the new fluid temperature for the process to be reversible. These processes therefore all require an infinity of thermal reservoirs.

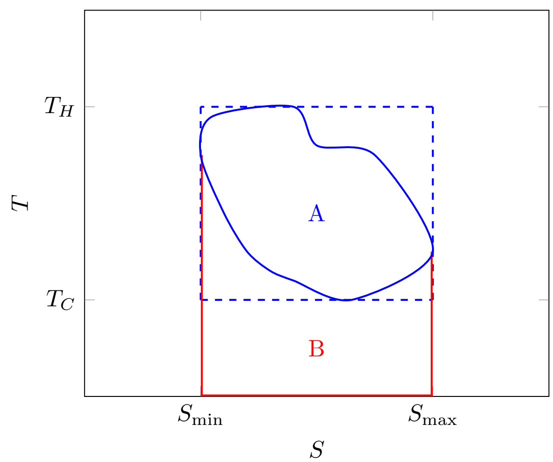
   We are left with only isothermal and isentropic processes. The only possible cycle that can be made with these processes is, by definition, the Carnot cycle, and we have thus shown that the only reversible cycle operating between two thermal reservoirs is the Carnot cycle.



(a) Cycle for question 1.



(b) Cycle for question 2.



(c) Cycle for question 3.