The Generalized Mean

Consider the generalized mean for non-negative x_i

$$M(p; x_1, \ldots, x_n) := \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p}$$

whose name becomes self-explanatory when we consider the six cases

$$M(-1; x_1, \ldots, x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-1} \right)^{-1} = \frac{n}{x_1^{-1} + \ldots + x_n^{-1}} \qquad \text{harmonic mean}$$

$$M(0; x_1, \ldots, x_n) = \lim_{p \to 0} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p} = (x_1 \cdots x_n)^{1/n} \qquad \text{geometric mean}$$

$$M(1; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{x_1 + \ldots + x_n}{n} \qquad \text{arithmetic mean}$$

$$M(2; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 = \left(\frac{x_1^2 + \ldots + x_n^2}{n} \right)^{1/2} \qquad \text{quadratic mean}$$

$$M(-\infty; x_1, \ldots, x_n) = \lim_{p \to -\infty} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p} = \min\{x_1, \ldots, x_n\} \qquad \text{minimum}$$

$$M(\infty; x_1, \ldots, x_n) = \lim_{p \to \infty} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p} = \max\{x_1, \ldots, x_n\} \qquad \text{maximum}$$

The three cases involving limits are somewhat more difficult to evaluate. In these cases, the calculations are simplified by first considering the natural logarithm of the limit and hence applying L'Hôpital's rule.

$$\log L_0 = \lim_{p \to 0} \frac{\log((x_1^p + \ldots + x_n^p)/n)}{p} = \lim_{p \to 0} \frac{x_1^p \log x_1 + \ldots + x_n^p \log x_n}{x_1^p + \ldots + x_n^p} = \frac{\log x_1 \cdots x_n}{n}$$

$$\log L_\infty = \lim_{p \to \infty} \frac{1}{p} \log \frac{x_1^p + \ldots + x_n^p}{n} = \log x_k + \lim_{p \to \infty} \frac{1}{p} \log \frac{1}{n} \left[\left(\frac{x_1}{x_k} \right)^p + \ldots + 1 + \ldots + \left(\frac{x_n}{x_k} \right)^p \right] = \log x_k$$

$$L_{-\infty} = \left[M\left(\infty; \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) \right]^{-1} = \left[\max\left\{ \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right\} \right]^{-1} = \min\{x_1, \ldots, x_n\}$$

The first case is simple. In the second case we have chosen $x_k = \max\{x_1, \ldots, x_n\}$ in order that all terms $(x_i/x_k)^p$, $i \neq k$, tend to 0 as p tends to ∞; and we have sneakily shown the third case to be a special case of the second and thus have avoided needing to calculate a limit.

Generalized Mean Inequalities

From the six examples of the generalized mean above, it seems plausible that M monotonically increases with p. In order to prove this rigorously, we must show that the partial derivative of M with respect to p is non-negative for all p. As with the limits earlier, our calculations may be simplified if we instead consider $\log M$. We obtain

$$\frac{\partial \log M}{\partial p} = \frac{\partial}{\partial p} \frac{\log(x_1^p + \ldots + x_n^p) - \log n}{p} = \frac{1}{p} \frac{x_1^p \log x_1 + \ldots + x_n^p \log x_n}{x_1^p + \ldots + x_n^p} - \frac{\log x_1^p + \ldots + \log x_n^p}{p^2} + \frac{\log n}{p^2}$$

Now normalizing the x_i^p by substituting $x_i^p \to x_i^p/(x_1^p + \ldots + x_n^p)$ results in $x_1^p + \ldots + x_n^p = 1$ and a simplification of our partial derivative to

$$\frac{x_1^p \log x_1 + \ldots + x_n^p \log x_n}{p} + \frac{\log n}{p^2} = \frac{1}{p^2} \left(x_1^p \log x_1^p + \ldots + x_n^p \log x_n^p + \log n \right)$$

In order to continue with our analysis, we will first prove *Jensen's inequality*.

Theorem. *For a convex function f with $c_1 + \cdots + c_n = 1$, $f(c_1x_1 + \cdots + c_nx_n) \leq c_1f(x_1) + \cdots + c_nf(x_n)$. For a concave function, the inequality is reversed.*

Proof. Compare f and its tangent line g at $c_1x_1 + \cdots + c_nx_n$. Then $f(c_1x_1 + \cdots + c_nx_n) = g(c_1x_1 + \cdots + c_nx_n) = c_1g(x_1) + \cdots + c_ng(x_n) \leq c_1f(x_1) + \cdots + c_nf(x_n)$, where the first equality follows from the equivalence of the function and the tangent line at their point of tangency, the second from linearity, and the third from convexity. □

Remark. While a "linear" function $g(x) = ax + b$ is not usually linear, the condition $c_1 + \cdots + c_n = 1$ allows us to write $g(c_1x_1 + \cdots + c_nx_n) = a(c_1x_1 + \cdots + c_nx_n) + 1b = a(c_1x_1 + \cdots + c_nx_n) + (c_1 + \cdots + c_n)b = c_1ax + c_1b + \cdots + c_nax_n + c_nb = c_1g(x_1) + \cdots + c_ng(x_n)$, establishing linearity.

Apply Jensen's inequality on the function $f(x) = x \log x$, $x \in (0, 1]$, letting $c_i = 1/n$ and $f(x_i) = x_i^p$ for all i. Since $f''(x) = 1/x$ is convex on the interval $(0, 1]$, we have the inequality

$$\begin{aligned}
f\left(\frac{1}{n}\right) = f\left(\frac{x_1^p}{n} + \ldots + \frac{x_n^p}{n}\right) &\leq \frac{f(x_1^p)}{n} + \ldots + \frac{f(x_n^p)}{n} \\
\frac{1}{n}\log\frac{1}{n} &\leq \frac{1}{n}(x_1^p \log x_1^p + \ldots + x_n^p \log x_n^p) \\
-\log n &\leq x_1^p \log x_1^p + \ldots + x_n^p \log x_n^p \\
0 &\leq x_1^p \log x_1^p + \ldots + x_n^p \log x_n^p + \log n
\end{aligned}$$

Hence, by our above expression for the partial derivative of M with respect to p, we have established that

$$\frac{\partial M}{\partial p} = \frac{1}{p^2}\left(x_1^p \log x_1^p + \ldots + x_n^p \log x_n^p + \log n\right) \geq 0$$

and that M is a monotonically increasing function. We have thus also established the inequalities

$$\min(\mathbf{x}) \leq \mathrm{HM}(\mathbf{x}) \leq \mathrm{GM}(\mathbf{x}) \leq \mathrm{AM}(\mathbf{x}) \leq \mathrm{QM}(\mathbf{x}) \leq \max(\mathbf{x}) \text{ for } \mathbf{x} := x_1, \ldots, x_n.$$