Electric Fields in Matter (Griffiths Ch. 4)

Matter is broadly categorized into conductors and insulators (dielectrics). Insulators are polarized in an electric field: neutral atoms deform into dipoles, experiencing a force $(\mathbf{p}\cdot\nabla)\mathbf{E}$, a torque $\mathbf{p}\times\mathbf{E}$ and having energy $-\mathbf{p}\cdot\mathbf{E}$. Define the polarization $\mathbf{P}$ as the dipole moment per unit volume; the potential caused by the polarized object is equivalent to that of an (otherwise neutral) object with surface charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ and volume charge $\rho_b = -\nabla\cdot\mathbf{P}$. Equivalently, we may consider a polarized object as the superposition of two oppositely charged objects, slightly displaced with respect to one another.

It is convenient to define the electric displacement $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, an analogue to the electric field $\mathbf{E}$, when discussing electric fields in matter, since Gauss's law applied to $\mathbf{D}$ reads $\nabla\cdot\mathbf{D} = \rho_f$, and all our tricks with Gaussian surfaces carry over. On the other hand, $\mathbf{D}$ is not a perfect analogue to $\mathbf{E}$: a vector field is determined by its divergence and curl, and while we have $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$, $\nabla\times\mathbf{E} = 0$, and $\nabla\cdot\mathbf{D} = \rho_f$, it is not necessarily true that $\nabla\times\mathbf{D} = 0$. When the latter is true, then the analogy can indeed simplify problems. [In general, we find that $\nabla\times\mathbf{D} = \nabla\times\mathbf{P} = 0$ fails at boundaries unless $\mathbf{P}$ is orthogonal to the surface, for in that case $\mathbf{P}\cdot\mathrm{d}\mathbf{l} = 0$.]

We may also rewrite our boundary conditions for $\mathbf{E}$ in terms of $\mathbf{D}$, which may be more convenient to apply:

$$\begin{aligned} E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} &= \frac{\sigma}{\epsilon_0} &\iff\quad& D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \\ \mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} &= 0 &\iff\quad& \mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}} \end{aligned}$$

Especially important are linear dielectrics, those for which the polarization may be written $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, and the electric displacement $\mathbf{D} = \epsilon_0(1+\chi_e)\mathbf{E} =: \epsilon\mathbf{E}$. We call χ_e the electric susceptibility, ϵ the permittivity of the material, and $\epsilon_r = 1+\chi_e$ the relative permittivity or dielectric constant of the material. In the case where $\nabla\times\mathbf{D} = 0$, comparing $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$ and $\nabla\cdot\mathbf{D} = \rho_f$ yields simply $\mathbf{D} = \epsilon_0\mathbf{E}_{\text{vac}}$ and hence $\mathbf{E} = \mathbf{D}/\epsilon = \mathbf{E}_{\text{vac}}/\epsilon_r$.

In a homogeneous linear dielectric, the bound charge density ρ_b is proportional to the free charge density ρ_f:

$$\rho_b = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left(\epsilon_{\mathbf{0}}\frac{\chi_{\mathbf{e}}}{\mathbf{e}}\mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f,$$

and, in particular, unless free charge is embedded in the material, $\rho_f = \rho_b = 0$, and any net charge must reside at the surface. In this case, the potential within the dielectric satisfies Laplace's equation, and we can generalize electrostatic boundary value problems to include linear dielectrics.

The linear-dielectric analogue of energy stored in an electric system is

$$U = \frac{1}{2}\epsilon_0\int E^2\,\mathrm{d}\tau \to \frac{1}{2}\epsilon\int E^2\,\mathrm{d}\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,\mathrm{d}\tau.$$

Consider a situation in which a dielectric slab is partway within a capacitor. The force pulling on the dielectric slab is given by

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q^2}{2C}\right) = \frac{1}{2}\frac{Q^2}{C^2}\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{1}{2}V^2\frac{\mathrm{d}C}{\mathrm{d}x},$$

where we use the fact that the energy stored in a capacitor is $CV^2/2 = Q^2/2C$ and consider the scenario in which Q is held fixed as the force acts on the dielectric slab. [Because the force acting on the slab is purely dictated by the charge configuration at any point, it makes no difference how the charge configuration is changing—the scenario in which Q is held fixed is simply easiest computationally.]

Magnetostatics (Griffiths Ch. 5)

The Lorentz force law, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, provides the equation of motion in electro- and magnetostatics. It is convenient to rewrite the Lorentz force law in terms of currents via the substitution $q\mathbf{v} \to \mathbf{I}l$, yielding

$$\mathbf{F}_{\text{mag}} = I \int \mathrm{d}\mathbf{l} \times \mathbf{B}.$$

We also frequently encounter surface and volume currents; those suggest the substitutions $\mathbf{I}\,\mathrm{d}l \to \mathbf{K}\,\mathrm{d}a$, $\mathbf{I}\,\mathrm{d}l \to \mathbf{J}\,\mathrm{d}\tau$, where $\mathbf{K}$ and $\mathbf{J}$ are the surface and volume current densities respectively.

The magnetic field of a steady line current is given by the Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}''}{r''^2}\,\mathrm{d}l' = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\mathbf{l}' \times \hat{\mathbf{r}}''}{r''^2}.$$

The Biot-Savart law yields the two Maxwell equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. The latter is Ampere's law; in integral form,

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}}.$$

Symmetry arguments and the Biot-Savart law often permit us to identify the direction of the magnetic field; Ampere's law can often do the rest.

That $\nabla \cdot \mathbf{B} = 0$ permits us to identify a vector potential $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$. Since the curl is unique up to an additive gradient of a scalar, we have the freedom to also set $\nabla \cdot \mathbf{A} = 0$, the Coulomb gauge. Ampere's law then becomes

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \quad \Longleftrightarrow \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

This is Poisson's equation; assuming $\mathbf{J}$ goes to 0 at infinity, we can read off the solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r''}\,\mathrm{d}\tau'.$$

If the current configuration is infinite, then other means must be found; one such uses

$$\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a} = \Phi$$

and exploits the analogy between the above equation and Ampere's law. For example, finding the magnetic potential of an infinite solenoid is analogous to finding the magnetic field of a fat wire: knowing the direction of the magnetic field of the wire gives us the direction of the magnetic potential of the solenoid, and the above equation does the rest. Another alternative is to match the components of $\nabla \times \mathbf{A} = \mathbf{B}$, noting that in most circumstances the magnetic potential is in the same direction as the current.

The Maxwell equations provide also magnetostatic boundary conditions:

$$\int \mathbf{B}\cdot d\mathbf{a} = 0 \quad \Longleftrightarrow \quad B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad \Longleftrightarrow \quad B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = \mu_0 K.$$

Similarly, $\nabla\cdot\mathbf{A} = 0$ and $\nabla\times\mathbf{A} = \mathbf{B}$ yield $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$ and

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.$$

The magnetic dipole moment is the dominant term in the multipole expansion of the vector potential (since the monopole term is identically zero) and is given by

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}, \ \mathbf{m} = I\mathbf{a}.$$

Magnetic Fields in Matter (Griffiths Ch. 6)

Magnetic phenomena arise from charges in motion. Electrons orbiting nuclei and spinning about their axes (in a classical picture) can be treated as magnetic dipoles, experiencing a force $\nabla(\mathbf{m}\cdot\mathbf{B})$, a torque $\mathbf{m}\times\mathbf{B}$ and having energy $-\mathbf{m}\cdot\mathbf{B}$ in an applied magnetic field. Paramagnetism, the attraction of objects to magnetic fields, arises from the net torque on these microscopic dipoles in atoms and molecules with unpaired electrons. In atoms and molecules with paired electrons, the torques on each electron-dipole generally cancel, resulting in no net torque.

Although the formulae for electric and magnetic dipoles are similar, they are very different in origin. Electric dipoles consist of paired positive and negative charges, whereas magnetic dipoles do not—magnetic monopoles do not exist. Instead, magnetic dipoles are caused by (microscopic) current loops. The differences between the two types of dipoles is most evident close to the dipole itself: the direction of the electric field between the two charges is *opposite* from that of the electric field outside, whereas the direction of the magnetic field at the center of the current loop is *the same* as that of the magnetic field outside.

An applied magnetic field also retards the period of atomic orbits and creates a magnetic dipole moment opposing the field, resulting in diamagnetism, the repulsion of objects from magnetic fields. Diamagnetism is a universal phenomenon; however, being much weaker than paramagnetism, diamagnetism is usually observable only in atoms and molecules with paired electrons. Define the magnetization $\mathbf{M}$ as the magnetic dipole moment per unit volume, analogous to the polarization $\mathbf{P}$. The vector potential resulting from this magnetization is equivalent to that of a surface current $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$ and that of a volume current $\mathbf{J}_b = \nabla\times\mathbf{M}$.

It is convenient to define an auxiliary field $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, an analogue to the magnetic field $\mathbf{B}$, when discussing magnetic fields in matter, since Ampere's law applied to $\mathbf{H}$ reads $\nabla\times\mathbf{H} = \mathbf{J}_f$, and all our tricks with Amperian loops carry over. On the other hand, $\mathbf{H}$ is not a perfect analogue to $\mathbf{B}$: a vector field is determined by its divergence and curl, and while we have $\nabla\cdot\mathbf{B} = 0$, $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$, and $\nabla\times\mathbf{J} = \mathbf{J}_f$, it is not necessarily true that $\nabla\cdot\mathbf{H} = 0$. When the latter is true, then the analogy can indeed simplify problems. [In general, we find that $\nabla\cdot\mathbf{H} = -\nabla\cdot\mathbf{M} = 0$ fails at boundaries unless $\mathbf{M}$ is tangent to the surface, for in that case $\mathbf{M}\cdot d\mathbf{a} = 0$.]

We may also rewrite our boundary conditions for $\mathbf{B}$ in terms of $\mathbf{H}$, which may be more convenient to apply:

$$\nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{M} \qquad \Longleftrightarrow \qquad H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f \, (\mathbf{K}_f) \qquad \Longleftrightarrow \qquad \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

Especially important are linear paramagnets and diamagnets, those for which the magnetization may be written $\mathbf{M} = \chi_m \mathbf{H}$, and the magnetic field $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} =: \mu\mathbf{H}$. We call χ_m the magnetic susceptibility and μ the permeability of the material. In the case where $\nabla \cdot \mathbf{H} = 0$, comparing $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and $\nabla \times \mathbf{H} = \mathbf{J}_f$ yields simply $\mathbf{H} = \mathbf{B}_{\text{vac}}/\mu_0$ and hence $\mathbf{B} = \mu\mathbf{H} = (\mu/\mu_0)\mathbf{B}_{\text{vac}}$.

In a homogeneous linear paramagnet or diamagnet, the (volume) bound current density $\mathbf{J}_b$ is proportional to the free current density $\mathbf{J}_f$:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times \chi_{\mathbf{m}} \mathbf{H} = \chi_m \mathbf{J}_f,$$

and, in particular, unless free current flows through the material, $\mathbf{J}_f = \mathbf{J}_b = 0$, and any net current must reside at the surface. If $\mathbf{J}_f = 0$ everywhere, then the curl of $\mathbf{H}$ vanishes and $\mathbf{H}$ can be expressed as the gradient of a scalar potential W, $\mathbf{H} = -\nabla W$, with $\nabla^2 W = -\nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{M}$. This is Poisson's equation, and we can generalize electrostatic boundary value problems to include linear paramagnets and diamagnets.

Electro- and magnetostatics share surprising parallels. On certain occasions, solutions to an electrostatic problem can be transcribed directly into solutions to a magnetostatic problem. Consider, for example, the following:

$\nabla \cdot \mathbf{D} = 0$	$\nabla \times \mathbf{E} = 0$	$\epsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$	no free charge
$\nabla \cdot \mathbf{B} = 0$	$\nabla \times \mathbf{H} = 0$	$\mu_0 \mathbf{H} = \mathbf{B} - \mu_{\mathbf{0}} \mathbf{M}$	no free current

The transcription $\mathbf{D} \to \mathbf{B}$, $\mathbf{E} \to \mathbf{H}$, $\mathbf{P} \to \mu_{\mathbf{0}}\mathbf{M}$, $\epsilon_0 \to \mu_0$ turns an electrostatic problem into a magnetostatic one. Similarly, consider

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \, \frac{\mathbf{r}''}{r''^2} \, \mathrm{d}\tau'$$
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \mathbf{P}(\mathbf{r}') \cdot \frac{\mathbf{r}''}{r''^2} \, \mathrm{d}\tau'$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{M}(\mathbf{r}') \times \frac{\mathbf{r}''}{r''^2} \, \mathrm{d}\tau'.$$

If ρ, $\mathbf{P}$ and $\mathbf{M}$ are uniform, then the same integral is involved in all three, and knowing the solution to one of the three equations above immediately allows the solutions to the other two equations to be written down.

Electrodynamics (Griffiths Ch. 7)

For most substances, the current density $\mathbf{J}$ is proportional to the force per unit charge, $\mathbf{f}$: $\mathbf{J} = \sigma\mathbf{f} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \approx \sigma\mathbf{E}$. This is Ohm's law; σ is the conductivity of the material, and $\rho = 1/\sigma$ the resistivity.

There are two forces that drive current around a circuit: the source $\mathbf{f}_s$, normally confined to a section of the circuit, like a battery, and the electrostatic force, which smooths out the flow of current. Together, we have $\mathbf{f} = \mathbf{f}_s + \mathbf{E}$. The net effect of this force about one loop of the circuit is the electromotive force $\mathcal{E}$,

$$\mathcal{E} := \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{f}_s \cdot \mathrm{d}\mathbf{l}.$$

In an ideal source of electromotive force, like a resistanceless battery, an infinitesimal net force is enough to start the charges moving, $\mathbf{f} = 0 = \mathbf{f}_s + \mathbf{E}$, and the potential difference between the terminals a and b is

$$V = -\int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_a^b \mathbf{f}_s \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{f}_s \cdot \mathrm{d}\mathbf{l} = \mathcal{E}.$$

Batteries are likely the most familiar source of electromotive force, but the most common uses generators. Generators make use of motional emf, which arise when wires, say, are moved through magnetic fields. Charges in the wire experience a magnetic force that drives current around the loop, and the emf is then

$$\mathcal{E} = \oint \mathbf{f}_{\mathrm{mag}} \cdot \mathrm{d}\mathbf{l}.$$

There is a handy flux rule applicable to moving loops through magnetic fields, $\mathcal{E} = -\mathrm{d}\Phi/\mathrm{d}t$, which simplifies some calculations, but has limited applicability outside of current loops.

In general, we must make corrections (or restore terms not present in statics) to Maxwell's equations when dealing with moving charges. Changing magnetic fields induce electric fields as dictated by Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

and our flux rule can be extended to any situation in which the magnetic flux through a current loop changes. In a pure Faraday field, we have $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0 = 0$, since the field is completely caused by induction, analogous to the equations for $\mathbf{B}$ in the magnetostatic regime, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and $\nabla \cdot \mathbf{B} = 0$. Faraday-induced electric fields are determined by $-\partial \mathbf{B}/\partial t$ in exactly the same way that magnetostatic fields are determined by $\mu_0 \mathbf{J}$, and it follows that all the tricks with Ampere's law can be used here.

Consider a scenario with two loops of wire. When current is run through one loop, the magnetic field generated changes the magnetic flux through the other loop of wire, and by the Biot-Savart law we see that the magnetic field, and hence the magnetic flux, is proportional to the current run through the first loop, and we can then write $\Phi_2 = M_{21} I_1$, with M_{21} the mutual inductance between the two loops. It turns out that $M_{21} = M_{12} = M$, that is, the mutual inductance is the same whether you run current through the first loop or through the second. Further, M depends solely on the geometry between the two current loops. Actually, running current through a loop not only induces an electromotive force in other loops, but also induces one in the source loop, $\Phi = LI$. Here we call L the self-inductance of the loop, and can write the induced emf as $\mathcal{E} = -L(\mathrm{d}I/\mathrm{d}t)$.

The total energy stored in a magnetic field is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\mathcal{E} I = LI \frac{\mathrm{d}I}{\mathrm{d}t} \quad \Longleftrightarrow \quad U = \frac{1}{2} L I^2.$$

We can rewrite this in an interesting form, much as we did in the case of electric fields:

$$U = \frac{1}{2\mu_0} \int B^2 \, \mathrm{d}\tau.$$

Much as we re-inserted terms into Maxwell's equations to reach Faraday's law, we must similarly re-insert terms to fix Ampere's law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

as can be concluded from conservation of charge. This provides a pleasing symmetry: just as changing magnetic fields induce electric fields, changing electric fields induce magnetic fields. The extra term in Ampere's law is called the displacement current, $\mathbf{J}_d = \epsilon_0 \partial \mathbf{E}/\partial t$. We finally arrive at the complete set of Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

When dealing with matter, we must introduce also the polarization current, with current density $\mathbf{J}_p = \partial \mathbf{P}/\partial t$, which comes from the current due to movement of charges when the polarization is changed. Rewriting Maxwell's equations in terms of free charges and currents, we obtain

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Here the second term in the Ampere's law equation is called the displacement current. The boundary conditions are identical to those derived earlier, for the additional introduced terms involve fluxes that vanish in the limit of the infinitesimal loops used to identify boundary conditions. In particular, if there is no free charge or current at the interface,

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = 0 \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0$$

$$B_1^\perp - B_2^\perp = 0 \qquad \frac{\mathbf{B}_1^\parallel}{\mu_1} - \frac{\mathbf{B}_2^\parallel}{\mu_2} = 0.$$

Conservation Laws (Griffiths Ch. 8)

Local conservation of charge gives us the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.$$

A charge and current distribution is known at an initial time. In the next instant, the charges and currents move: what is the work done by the electromagnetic forces on this charge and current distribution? From the Lorentz force law, $\mathbf{F} = q\mathbf{E} + \mathbf{v} \times \mathbf{B}$, we obtain

$$dW = \mathbf{F} \cdot \mathrm{d}\mathbf{l} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,\mathrm{d}t = q\mathbf{E} \cdot \mathbf{v}\,\mathrm{d}t.$$

Passing to the continuum limit, so that $q \to \rho\,\mathrm{d}\tau$ and $\rho\mathbf{v} \to \mathbf{J}$, followed by applying Maxwell's equations and vector identities, we find

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_V \mathbf{E} \cdot \mathbf{J}\,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \mathrm{d}\tau - \frac{1}{\mu_0} \oint_S (\mathbf{E} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a}.$$

This is Poynting's theorem; we can interpret the first term as the rate of change of the energy stored in the fields and the second as the flux passing through the surface. The energy flux density $\mathbf{S} := (\mathbf{E} \times \mathbf{B})/\mu_0$ is called the Poynting vector. We can rewrite this equation in differential form, expressing local conservation of energy:

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S}.$$

Consider the same distribution as before: what is the total electromagnetic force on the charge and current distribution in a volume V? From the Lorentz force law, and again passing to the continuum limit, we obtain the force per unit volume $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$. Introducing Maxwell's stress tensor T,

$$T_{ij} := \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right),$$

we find that the total force becomes, upon applying the divergence theorem,

$$\mathbf{F} = \oint_S \mathsf{T} \cdot \mathrm{d}\mathbf{a} - \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathbf{S}\, \mathrm{d}\tau.$$

Physically, T represents the force per unit area (stress) acting on the surface: its diagonal terms are pressures, and its off-diagonal terms shears. By Newton's second law,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_V \epsilon_0 \mu_0 \mathbf{S}\, \mathrm{d}\tau + \oint_S \mathsf{T} \cdot \mathrm{d}\mathbf{a}.$$

Once again, this admits a physical interpretation: the first term represents momentum stored in the fields, and the second the momentum flux through the surface. We can also rewrite this equation in differential form, expressing local conservation of momentum:

$$\frac{\partial}{\partial t}(\pi_{\text{mech}} + \pi_{\text{em}}) = \nabla \cdot \mathsf{T}.$$

Using $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, we can treat angular momentum similarly.