## Maximum work from N objects

Find the maximum work that can be extracted from N identical objects with initial temperatures  $T_1, \ldots, T_N$ . Each object has constant heat capacity  $C_V = C$ .

SOLUTION 1. (Pure math.) Our physical problem can be recast as maximizing

$$W = C \sum_{i} (T_i - T'_i)$$
 subject to the constraint  $\epsilon = C \sum_{i} \ln\left(\frac{T'_i}{T_i}\right), \quad \epsilon \ge 0.$ 

The first equality follows from the first law of thermodynamics. Heat is exchanged solely between objects within the system and does not leave the system, so any change in the internal energy must be due to work. The second "equality" follows from the second law of thermodynamics, *i.e.* entropy is non-decreasing. This is really an inequality masquerading as an equality, but writing it in this form makes the following manipulations more intuitive.  $\epsilon$  is a non-negative parameter we can freely modify, since the second law is satisfied for any non-negative  $\epsilon$ .

We will apply the technique of Lagrange multipliers. Some algebra lets us write the Lagrangian as

$$\mathcal{L}(\{T'_i\}, \lambda) = -\sum_i T'_i + \lambda \left(\prod_i T'_i - \epsilon' \prod_i T_i\right), \quad \epsilon' \ge 1.$$

All additive and multiplicative constants have been removed, which is permissible because this does not affect any critical points of the Lagrangian. Alternatively, one can leave them in and note that they vanish upon setting partial derivatives of the Lagrangian equal to 0. We have

$$0 = \frac{\partial \mathcal{L}}{\partial T'_j} = -1 + \frac{\lambda}{T'_j} \prod_i T'_i = -1 + \frac{\epsilon' \lambda}{T'_j} \prod_i T_i \quad \Longleftrightarrow \quad T'_j = \text{constant},$$

since all other quantities in the right-most term of the first equation are constants. The third equality follows from the second-law constraint, or formally from  $0 = \partial \mathcal{L}/\partial \lambda$ . In addition, since W is maximal when  $T'_j$  is smallest and  $T'_j$  is proportional to  $\epsilon'$ , we set  $\epsilon' = 1$  to maximize W. By our second-law constraint, this yields

$$T'_{j} = \left(\prod_{i} T_{i}\right)^{1/N} \quad \Longleftrightarrow \quad W = C\left(\sum_{i} T_{i} - N\left(\prod_{i} T_{i}\right)^{1/N}\right) = NC\left(\frac{1}{N}\sum_{i} T_{i} - \left(\prod_{i} T_{i}\right)^{1/N}\right).$$

The right-most term is interesting, because the quantity within the brackets is exactly the difference of the arithmetic and geometric means of the initial temperatures.

SOLUTION 2. (Some thermodynamics.) Two known facts from thermodynamics solve the maximization problem completely. First, maximum work is accomplished for a reversible process, for which the net entropy change, and hence  $\epsilon$ , is zero. One can show that this corresponds to  $\epsilon' = 1$ . Furthermore, all final temperatures  $\{T'_i\}$  must be equal, for otherwise we can extract more work from the system by running a Carnot engine between any two objects with a temperature differential. The remainder of the solution proceeds as before.