Symmetry. We call a transformation a *symmetry* if it (i) is structure-preserving, (ii) is a C^∞ diffeomorphism (a smooth invertible mapping with a smooth inverse), and (iii) maps the object to itself. To clarify (i), under a given transformation, a rigid structure must remain rigid, but a flexible structure can become warped.

Lie symmetries. The set of symmetries Γ_ϵ,

$$\Gamma_\epsilon : x^s \mapsto \hat{x}^s(x^1, \cdots, x^N; \epsilon), \quad s = 1, \cdots, N,$$

is a *one-parameter local Lie group* if (i) Γ_0 is the trivial symmetry, (ii) Γ_ϵ is a symmetry for all ϵ in some ϵ-neighborhood of 0, (iii) $\Gamma_\delta \Gamma_\epsilon = \Gamma_{\delta+\epsilon}$ for all δ, ϵ in some ϵ-neighborhood of 0, and (iv) each of the $\hat{x}^s$ can be represented as a Taylor series in ϵ for all ϵ in some ϵ-neighborhood of 0:

$$\hat{x}^s(x^1, \cdots, x^N; \epsilon) = x^s + \epsilon \xi^s(x^1, \cdots, x^N) + O(\epsilon^2), \quad s = 1, \cdots, N.$$

In particular, (iii) tells us that $\Gamma_{-\epsilon} = \Gamma_\epsilon^{-1}$. Lie symmetries must depend continuously on some parameter ϵ, unlike discrete symmetries, which do not. An n-parameter Lie group can be regarded as a composition of n one-parameter Lie groups. In what follows, we will consider the qualifier *local* implicit and adopt the notation $x^1 = x$, $x^2 = y$, etc.

Symmetries for first-order ODEs. Given a first-order ODE $y' = \omega(x, y)$, the symmetry condition is $\hat{y}' = \omega(\hat{x}, \hat{y})$. Considering $\hat{x} = \hat{x}(x, y)$ and $\hat{y} = \hat{y}(x, y)$, we find by differentiation

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} = \frac{\mathrm{d}\hat{y}/\mathrm{d}x}{\mathrm{d}\hat{x}/\mathrm{d}x} = \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} = \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}),$$

where the last equality imposes the symmetry condition. A solution curve that maps to itself under a symmetry is called *invariant*. If all solution curves are invariant under a symmetry, then the symmetry is called *trivial*.

Solving first-order ODEs with Lie symmetries. Suppose that translational symmetry in y exists: $(\hat{x}, \hat{y}) = (x, y + \epsilon)$. By the symmetry condition on y, we have $\omega(x, y) = \omega(\hat{x}, \hat{y}) = \omega(x, y + \epsilon) = \omega(x)$, since ϵ is arbitrary in some ϵ-neighborhood of 0. Then, at least in such a neighborhood,

$$y' = \omega(x, y) = \omega(x) \quad \Longleftrightarrow \quad y = \int \omega(x)\,\mathrm{d}x + c.$$

It turns out that all one-parameter Lie groups can be represented as translations in a suitable coordinate system, as discussed below.

Action of Lie symmetries. The *action* of a symmetry on the xy-plane is the mapping from (x, y) to $(\hat{x}, \hat{y}) = (\hat{x}(x, y), \hat{y}(x, y))$. Under this action, the solution curves $\{(x, f(x))\}$ are mapped to $\{(\hat{x}, \tilde{f}(\hat{x}))\}$, which defines the function $\tilde{f}$. The solution curve $y = f(x)$ is invariant under the symmetry if $f = \tilde{f}$.

Orbits. The *orbit* of a Lie symmetry through (x, y) is the set of points $\{(\hat{x}, \hat{y})\}$ to which (x, y) can be mapped by variation of ϵ; $(\hat{x}, \hat{y}) = (\hat{x}(x, y; \epsilon), \hat{y}(x, y; \epsilon))$. If an orbit consists of a single point, we call that point an *invariant point*. By construction, every orbit is invariant under the action of the Lie group. Orbits cannot intersect, for otherwise the action of the Lie group at the intersection is not well-defined—which orbit would the point follow?

Invariant points. Consider the action of a Lie symmetry through an arbitrary point $(\hat{x}, \hat{y}) = (x, y; \epsilon = 0)$—arbitrary because the origin of the transformation parametrized by ϵ is arbitrary. The tangent vector to the orbit at any point is given by

$$\left(\frac{\mathrm{d}\hat{x}}{\mathrm{d}\epsilon}, \frac{\mathrm{d}\hat{y}}{\mathrm{d}\epsilon}\right) =: (\xi(\hat{x}, \hat{y}), \eta(\hat{x}, \hat{y})),$$

and expanding $\hat{x}$ (resp. $\hat{y}$) in a Taylor series about x (resp. y) yields

$$\hat{x} = x + \epsilon \xi(x, y) + O(\epsilon^2), \quad \hat{y} = y + \epsilon \eta(x, y) + O(\epsilon^2).$$

An invariant point is independent of the parameter ϵ, so the point (x, y) is an invariant point if and only if $\xi(x, y) = \eta(x, y) = 0$.

Characteristic. An invariant curve C must be, by definition, a subset of an orbit. Equivalently, the tangent to C at every point $(x, y) \in C$ must be parallel to the tangent vector $(\xi(x, y), \eta(x, y))$. We express this condition mathematically by defining the *characteristic* $Q(x, y, y')$ such that

$$Q(x, y, y') := \eta(x, y) - y'\xi(x, y).$$

The aforementioned condition holds whenever $Q = 0$.

Reduced characteristic. On all solutions of $y' = \omega(x, y)$, the characteristic becomes

$$\bar{Q}(x, y) = Q(x, y, \omega(x, y)) = \eta(x, y) - \omega(x, y)\xi(x, y),$$

and we call $\bar{Q}$ the *reduced characteristic*. A solution curve $y = f(x)$ is invariant if $\bar{Q} \equiv 0$ over the curve; if $\bar{Q} \equiv 0$ over all y governed by the differential equation, the Lie symmetry is trivial.

Canonical coordinates. We have seen that the translational symmetry $(\hat{x}, \hat{y}) = (x, y + \epsilon)$ readily allows for solution of the differential equation $y' = \omega(x, y)$. It remains to find these coordinates for which translational symmetry exists; that is, coordinates $(r, s) = (r(x, y), s(x, y))$ such that $(\hat{r}, \hat{s}) = (r, s + \epsilon)$. At the arbitrary point $(\hat{r}, \hat{s}) = (r, s; \epsilon = 0)$, we must have

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}\hat{r}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \hat{r}_x(x, y)\,\left.\frac{\mathrm{d}x}{\mathrm{d}\epsilon}\right|_{\epsilon=0} + \hat{r}_y(x, y)\,\left.\frac{\mathrm{d}y}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = r_x(x, y)\,\xi(x, y) + r_y(x, y)\,\eta(x, y), \\ 1 &= \left.\frac{\mathrm{d}\hat{s}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \hat{s}_x(x, y)\,\left.\frac{\mathrm{d}x}{\mathrm{d}\epsilon}\right|_{\epsilon=0} + \hat{s}_y(x, y)\,\left.\frac{\mathrm{d}y}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = s_x(x, y)\,\xi(x, y) + s_y(x, y)\,\eta(x, y). \end{aligned}$$

The invertibility condition associated with these coordinates is $r_x s_y - r_y s_x \neq 0$. Such a pair of coordinates $r(x, y)$, $s(x, y)$ is called a pair of *canonical coordinates*. By our construction of these coordinates, we further find that the curves of constant r are invariant orbits under the corresponding Lie group, and therefore call r the *invariant canonical coordinate*.

Canonical coordinates are not well-defined at invariant points, for there $\xi = \eta = 0$ and the determining condition for s is not satisfied. They are, however, well-defined on some ϵ-neighborhood of any noninvariant point, for which ξ and η are not both zero. Canonical coordinates are also not unique: if r and s are a pair of canonical coordinates, then so are $F(r)$ and $s + G(r)$, where F and G are arbitrary smooth functions such that $F'(r) \neq 0$, as given by the invertibility condition.

Method of characteristics for canonical coordinates. We may solve the PDE

$$1 = s_x(x, y)\,\xi(x, y) + s_y(x, y)\,\eta(x, y)$$

using the method of characteristics. By considering $s = s(x(t), y(t))$, taking its total time derivative, and comparing the resulting equation with the given PDE, we find

$$\frac{\mathrm{d}s}{\mathrm{d}t} = s_x\frac{\mathrm{d}x}{\mathrm{d}t} + s_y\frac{\mathrm{d}y}{\mathrm{d}t} \quad\Longrightarrow\quad \mathrm{d}s = \frac{\mathrm{d}x}{\xi(x, y)} = \frac{\mathrm{d}y}{\eta(x, y)},$$

which defines a *characteristic curve* on the surface $s(x, y)$. We assume $\eta(x, y)$ and $\xi(x, y)$ nonzero; the corresponding equations for η or ξ zero can be worked out analogously. The case when both η and ξ are zero corresponds to an invariant point, for which canonical coordinates cannot be defined.

First integral of characteristic curves. A *first integral* or *integral of motion* of a first-order ODE $y' = f(x, y)$ is a nonconstant function $\phi(x, y)$ whose value is constant on any solution $y = y(x)$ of the given ODE. In physical applications, this constant value is known as a *conserved quantity*. We have

$$\phi(x, y(x)) = c \quad\Longleftrightarrow\quad \phi_x + \phi_y y' = 0,$$

and, by comparison of the above condition to the condition on r as a canonical coordinate, we see that r is the first integral of the ODE

$$y' = \frac{\eta(x, y)}{\xi(x, y)}, \quad \xi(x, y) \neq 0.$$

This resulting equation can also be derived from the cyclic chain rule. This ODE corresponds exactly to that defining the characteristic curves on s. Solving this differential equation gives $y = y(x; r)$, r the constant of integration, which is assumed invertible to yield $r = r(x, y)$. In general, this ODE is simpler to solve than our original ODE, but a notable exception is the case of a trivial symmetry. In this case, the derived ODE is exactly our original ODE, since for a trivial symmetry $\eta(x, y) = \omega(x, y)\xi(x, y)$, and Lie-theoretic methods present no simplification.

Solving for canonical coordinates. The previous section discusses how to find the canonical coordinate r. The canonical pair s is obtained by integrating the characteristic curve equation to yield

$$s(x, y(x; r)) = s(x; r) = \int \frac{\mathrm{d}x}{\xi(x, y(x; r))} \quad \Longrightarrow \quad s(x; r) = s(x; r(x, y)) = s(x, y).$$

The substitution $y = y(r, x)$ comes from the previous section. Note that r serves only the role of a parameter: we integrate over a characteristic curve, and r is constant over each of the characteristic curves.

Solving ODEs with Lie symmetries. The ODE $y' = \omega(x, y)$ reduces to quadrature under the transformed coordinates r and s. In particular, given $x = x(r, s)$ and $y = y(r, s)$, we find that

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{s_x + \omega(x, y)s_y}{r_x + \omega(x, y)r_y} =: \varphi(x, y) = \Phi(x(r, s), y(r, s)) =: \Phi(r, s).$$

By construction, however,

$$\Phi(r, s + \epsilon) = \Phi(\hat{r}, \hat{s}) = \frac{\mathrm{d}\hat{s}}{\mathrm{d}\hat{r}} = \frac{\mathrm{d}s}{\mathrm{d}r} = \Phi(r, s).$$

The first equality follows by definition of $\hat{r}$ and $\hat{s}$, the second from the symmetry condition, and the third by explicit evaluation. Since ϵ is arbitrary, $\Phi(r, s) = \Phi(r)$, and we thus have

$$s(r) = \int \Phi(r)\,\mathrm{d}r + C,$$

which can be inverted as a function of x and y by hypothesis.

Linearized symmetry condition. We have previously found that

$$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} = \omega(\hat{x}, \hat{y}).$$

Expanding $\hat{x}$ (resp. $\hat{y}$) about x (resp. y) to first order, we have

$$\omega + \epsilon[\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y] = \frac{\epsilon\eta_x + \omega(1 + \epsilon\eta_y)}{1 + \epsilon\xi_x + \omega(\epsilon\xi_y)} = \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}) = \omega + \epsilon(\xi\omega_x + \eta\omega_y).$$

Hence simplifying yields the *linearized symmetry condition*,

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 = \xi\omega_x + \eta\omega_y.$$

This computation linearizes our original nonlinear PDE, making it more amenable to solution.