10. Tangent and normal plane.

Let C be an arbitrary curve in $\mathbb{R}^3$ and $\mathbf{x}(s)$ be a parametric representation of C with arclength s as parameter. Two points of C, corresponding to the values s and $s+h$ of the parameter, determine a chord of C with direction given by the vector $\mathbf{x}(s+h)-\mathbf{x}(s)$, and hence also by the vector $(\mathbf{x}(s+h)-\mathbf{x}(s))/h$. The vector

$$\mathbf{t}(s) = \lim_{h\to 0} \frac{\mathbf{x}(s+h)-\mathbf{x}(s)}{h} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \dot{\mathbf{x}}(s)$$

is called the unit tangent vector to the curve C at the point $\mathbf{x}(s)$.

$$|\mathbf{t}|^2 = \mathbf{t}\cdot\mathbf{t} = \mathbf{x}\dot{\cdot}\,\dot{x} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = 1,$$

so $\mathbf{t}$ is indeed a unit vector. The direction of $\mathbf{t}$ corresponds to increasing values of s and thus depends on the choice of the parametric representation; introducing any other allowable parameter t gets us

$$\mathbf{t}(t) = \dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t} = \frac{\mathbf{x}'}{\sqrt{(\mathbf{x}'\cdot\mathbf{x}')}} = \frac{\mathbf{x}'}{|\mathbf{x}'|},$$

since $\mathrm{d}s^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}$.

The straight line passing through a point P of C in the direction of the corresponding unit tangent vector is called the tangent to the curve C at P. The tangent can be represented in the form $\mathbf{y}(u) = \mathbf{x} + u\mathbf{t}$, where $\mathbf{x}$ and $\mathbf{t}$ depend on the point of C under consideration, and u is a real variable. We may also replace the unit vector $\mathbf{t}$ by any vector parallel to $\mathbf{t}$, say $\mathbf{x}'$, and hence obtain another parametric representation of the tangent, $\mathbf{y}(v) = \mathbf{x} + v\mathbf{x}'$. Straight lines are the only curves whose tangent direction is constant, as can be seen by integrating the vector equation $\mathbf{x}' = \mathbf{C}$.

The totality of all vectors bound at a point P of C which are orthogonal to the corresponding unit tangent vector lie in a plane, the normal plane to C at P.

11. Osculating plane.

Let $\mathbf{x}(t)$ be a parametric representation of a curve C. We seek the limit position of a plane E passing through three points P, P_1, and P_2 of C as P_1 and P_2 tend to P, the osculating plane of C at P.

Denote by t, $t+h_1$, and $t+h_2$ respectively the parametric values at P, P_1, and P_2. The chords PP_1 and PP_2 of C are given by the vectors $\mathbf{a}_i = \mathbf{x}(t+h_i) - \mathbf{x}(t)$; these vectors, if linearly independent, span the plane E. Any linear combination of these vectors is also linearly independent, and E is consequently also spanned by the vectors $\mathbf{v}^{(1)} = \mathbf{a}_1/h_1$ and $\mathbf{w} = 2(\mathbf{v}^{(2)} - \mathbf{v}^{(1)})/(h_2 - h_1)$, where $\mathbf{v}^{(2)}$ is defined like $\mathbf{v}^{(1)}$.

The choice of these unusual linear combinations is made clear upon a Taylor expansion of $\mathbf{x}(t)$,

$$\mathbf{x}(t+h_i) = \mathbf{x}(t) + h_i\mathbf{x}'(t) + \frac{h_i}{2}\mathbf{x}''(t) + \mathbf{o}(h_i^2),$$

where $\mathbf{o}(h_i^2)$ is a vector whose components are $o(h_i^2)$, and we consider a function $g(z) = o(f(z))$ if

$$\lim_{z\to 0}\frac{g(z)}{f(z)} = 0.$$

Hence we obtain

$$\mathbf{v}^{(1)} = \mathbf{x}'(t) + \frac{h_1}{2}\mathbf{x}''(t) + \mathbf{o}(h_1)$$
$$\mathbf{w} = \mathbf{x}''(t) + \mathbf{o}(1).$$

Consequently, if $h_i \to 0$, then $\mathbf{v}^{(1)} \to \mathbf{x}'(t)$ and $\mathbf{w} \to \mathbf{x}''(t)$. These vectors, if linearly independent, determine the osculating plane E of the curve C at P.

We denote by $\mathbf{z}$ the position vector of any point of the osculating plane O of C. The vector $\mathbf{z}-\mathbf{x}$ lies in O; the vectors $\mathbf{z}-\mathbf{x}$, $\mathbf{x}'$, and $\mathbf{x}''$ are linearly dependent. As such, we can represent the osculating plane of C at P as

$$|(\mathbf{z}-\mathbf{x})\;\mathbf{x}'\;\mathbf{x}''| = 0.$$

The osculating plane passes through the tangent, and the line formed by the intersection of the osculating plane with the corresponding normal plane is called the principal normal.

In the exceptional case that the vectors $\mathbf{x}'$ and $\mathbf{x}''$ are linearly dependent for all values of the parameter t, then the curve is necessarily a straight line. The osculating plane of a plane curve is always the plane of the curve.

12. Principal normal, curvature, osculating circle.

Let a curve C be given by an allowable parametric representation $\mathbf{x}(s)$ with arclength s as parameter. Differentiating the relation $\mathbf{t}\cdot\mathbf{t}=1$, we obtain $\mathbf{t}\cdot\dot{\mathbf{t}}=0$. Hence, if the vector $\dot{\mathbf{t}}=\ddot{\mathbf{x}}$ is not the null vector, it is orthogonal to the unit tangent vector $\mathbf{t}$ and consequently lies in the normal plane to C at the point under consideration; $\dot{\mathbf{t}}$ also lies in the osculating plane. The unit vector $\mathbf{p}(s)=\dot{\mathbf{t}}(s)/|\dot{\mathbf{t}}(s)|$, which has the direction and sense of $\dot{\mathbf{t}}$, is called the unit principal normal vector to C at $\mathbf{x}(s)$. The straight line passing through this point and containing $\mathbf{p}(s)$ is called the principal normal to C at this point.

The absolute value of the vector $\dot{\mathbf{t}}$,

$$\kappa(s) = |\dot{\mathbf{t}}(s)| = \sqrt{\ddot{\mathbf{x}}(s)\cdot\ddot{\mathbf{x}}(s)},$$

is called the curvature of C at $\mathbf{x}(s)$, whereas the reciprocal of the curvature, $\rho(s)=1/\kappa(s)$, is called the radius of curvature of C at $\mathbf{x}(s)$. Straight lines are the only curves whose curvature vanishes identically, because from $\kappa\equiv 0$ we obtain $\mathbf{x}(s)=\mathbf{a}(s)+\mathbf{c}$, with $\mathbf{a}$ and $\mathbf{c}$ constant vectors. For future convenience, we set $\mathbf{k}(s)=\dot{\mathbf{t}}(s)$, and call the vector $\mathbf{k}$ the curvature vector of C.

The direction of the unit tangent vector $\mathbf{t}$, being the first derivative of $\mathbf{x}$, depends on the orientation of the curve; the direction of the unit principal normal vector $\dot{\mathbf{t}}$, being the second derivative, does not, since replacing the parameter s by $-s$ introduces a negative sign to each derivative with respect to s.

The point M on the positive ray of the principal normal at distance $\rho(s)$ from the corresponding point P of C is called the center of curvature. The circle in the osculating plane whose radius is ρ and whose center is M is called the osculating circle or circle of curvature of C at P.

Let us derive an expression for the curvature κ under an arbitrary parameter t. First evaluating

$$\mathbf{x}' = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{t}\,\frac{\mathrm{d}s}{\mathrm{d}t}, \qquad \mathbf{x}'' = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{t}\,\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} + \mathbf{t}\,\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \kappa\mathbf{p}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + \mathbf{t}\,\frac{\mathrm{d}^2 s}{\mathrm{d}t^2},$$

and noting that $\mathbf{t}$ and $\mathbf{p}$ are orthogonal unit vectors, we find

$$\kappa = \frac{|\mathbf{x}'\times\mathbf{x}''|}{(\mathrm{d}s/\mathrm{d}t)^3} = \frac{|\mathbf{x}'\times\mathbf{x}''|}{|\mathbf{x}'|^3} = \frac{\sqrt{(\mathbf{x}'\cdot\mathbf{x}')(\mathbf{x}''\cdot\mathbf{x}'')-(\mathbf{x}'\cdot\mathbf{x}'')^2}}{(\mathbf{x}'\cdot\mathbf{x}')^{3/2}}.$$

13. Binormal. Moving trihedron of a curve.

To every point of the curve at which $\kappa>0$, we have associated two orthogonal unit vectors, the unit tangent vector $\mathbf{t}(s)$ and the unit principal normal vector $\mathbf{p}(s)$. We now introduce a third unit vector, $\mathbf{b}(s)=\mathbf{t}(s)\times\mathbf{p}(s)$, orthogonal to both $\mathbf{t}$ and $\mathbf{p}$, called the unit binormal vector of C at $\mathbf{x}(s)$. The straight line passing through $\mathbf{x}(s)$ in the direction of the corresponding unit binormal vector $\mathbf{b}(s)$ is called the binormal to C at $\mathbf{x}(s)$. The triple of vectors $\mathbf{t}$, $\mathbf{p}$, $\mathbf{b}$, in order, form a right-handed basis in $\mathbb{R}^3$, and is called the moving trihedron of the curve.

The vectors $\mathbf{t}$ and $\mathbf{p}$ span the osculating plane of the curve. The vectors $\mathbf{p}$ and $\mathbf{b}$ span the normal plane. The vectors $\mathbf{t}$ and $\mathbf{b}$ span the plane called the rectifying plane. Hence we have the following representations of each plane:

$$\begin{aligned} \text{Normal plane: } & (\mathbf{z}-\mathbf{x})\cdot\mathbf{t}=0 \\ \text{Rectifying plane: } & (\mathbf{z}-\mathbf{x})\cdot\mathbf{p}=0 \\ \text{Osculating plane: } & (\mathbf{z}-\mathbf{x})\cdot\mathbf{b}=0 \end{aligned}$$

14. Torsion.

The curvature $\kappa(s)$ measures the rate of change of the tangent when moving along the curve; more abstractly, it measures the deviation of the curve from a straight line in the neighborhood of any of its points. The torsion, to be introduced, measures the rate of change of the osculating plane when moving along the curve; more abstractly, it measures the magnitude and sense of deviation of a curve from the osculating plane in the neighborhood of the corresponding point of the curve.

The unit binormal vector $\mathbf{b}$ is normal to the osculating plane, and it is reasonable to expect that $\dot{\mathbf{b}}$ is a measure of the rate of change of the osculating plane and will be important in our investigations. Differentiating $\mathbf{b}\cdot\mathbf{t}=0$, we obtain

$$\dot{\mathbf{b}}\cdot\mathbf{t}=-\mathbf{b}\cdot\dot{\mathbf{t}}=-\kappa\mathbf{b}\cdot\mathbf{p}=0,$$

so $\dot{\mathbf{b}}$ and $\mathbf{p}$ are orthogonal. In addition, differentiating $\mathbf{b}\cdot\mathbf{b}=0$ shows that $\dot{\mathbf{b}}$ and $\mathbf{b}$ are orthogonal—so unless $\dot{\mathbf{b}}$ is the null vector, in which case the curve is a plane curve, $\dot{\mathbf{b}}$ must lie in the principal normal. We set $\dot{\mathbf{b}}(s)=-\tau(s)\mathbf{p}(s)$, or, upon taking the scalar product with $\mathbf{p}$ on both sides, $\tau(s)=-\mathbf{p}(s)\cdot\mathbf{b}(s)$, with τ called the torsion of C at $\mathbf{x}(s)$. The sign of the torsion is such that right-handed curves have a positive torsion.

Let us derive a representation of τ in terms of $\mathbf{x}(s)$ and its derivatives. Then

$$\tau=-\mathbf{p}\cdot\dot{\mathbf{b}}=-\mathbf{p}\cdot\frac{\mathrm{d}}{\mathrm{d}s}(\dot{\mathbf{x}}\times\mathbf{p})=-\mathbf{p}\cdot(\ddot{\mathbf{x}}\times\mathbf{p}+\dot{\mathbf{x}}\times\dot{\mathbf{p}}),$$

and, since $\mathbf{p}=\rho\ddot{\mathbf{x}}$, $\ddot{\mathbf{x}}\times\mathbf{p}=0$, and $\dot{\mathbf{p}}=\dot{\rho}\ddot{\mathbf{x}}+\rho\dddot{\mathbf{x}}$,

$$\tau=-|\rho\ddot{\mathbf{x}}\;\dot{\mathbf{x}}\;\dot{\rho}\ddot{\mathbf{x}}+\rho\dddot{\mathbf{x}}|=\rho^2|\dot{\mathbf{x}}\;\ddot{\mathbf{x}}\;\dddot{\mathbf{x}}|=\frac{|\dot{\mathbf{x}}\;\ddot{\mathbf{x}}\;\dddot{\mathbf{x}}|}{\ddot{\mathbf{x}}\cdot\ddot{\mathbf{x}}}.$$

The sign of the torsion is independent of the orientation of the curve, and hence it must have geometric significance. A Taylor expansion on $\mathbf{x}(s+h)$ gives

$$\mathbf{x}(s+h)=\mathbf{x}(s)+h\dot{\mathbf{x}}(s)+\frac{h^2}{2}\ddot{\mathbf{x}}(s)+\frac{h^3}{6}\dddot{\mathbf{x}}(s)+\mathbf{o}(h^3).$$

The vectors $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ lie in the osculating plane, so $h^3\dddot{\mathbf{x}}/6+\mathbf{o}(h^3)$ must determine the sense in which, in a neighborhood of the point $\mathbf{x}(s)$, the curve leaves the osculating plane. By continuity, the difference between this vector and $h^3\dddot{\mathbf{x}}(s)/6$ is arbitrarily small for sufficiently small values of h, and $\dddot{\mathbf{x}}(s)$ will therefore also determine the aforementioned sense. We will consider the torsion of right-hande curves. Recall that a curve is right-handed if, at $\mathbf{x}(s)$, for increasing values of s, it leaves the osculating plane to the positive side of the plane, as determined by the sense of the unit binormal vector $\mathbf{b}(s)$. In this case the angle between the vectors $\dddot{\mathbf{x}}$ and $\mathbf{b}=\dot{\mathbf{x}}\times\mathbf{p}=\dot{\mathbf{x}}\times\rho\ddot{\mathbf{x}}$ must be smaller than $\pi/2$. Since $\rho>0$, the same must be true for the angle between the vectors $\dddot{\mathbf{x}}$ and $\dot{\mathbf{x}}\times\ddot{\mathbf{x}}$. Hence $|\dot{\mathbf{x}}\;\ddot{\mathbf{x}}\;\dddot{\mathbf{x}}|>0$, and hence $\tau>0$. Right-handed curves thus have positive torsion, and a similar analysis shows the opposite for left-handed curves. The reciprocal of the torsion, $\sigma=1/\tau$, is called the radius of torsion.

Carefully differentiating, we can derive the following representation of τ for an arbitrary parameter t:

$$\tau=\frac{|\mathbf{x}'\;\mathbf{x}''\;\mathbf{x}'''|}{(\mathbf{x}'\times\mathbf{x}'')\cdot(\mathbf{x}'\times\mathbf{x}'')}=\frac{|\mathbf{x}'\;\mathbf{x}''\;\mathbf{x}'''|}{(\mathbf{x}'\cdot\mathbf{x}')(\mathbf{x}''\cdot\mathbf{x}'')-(\mathbf{x}'\cdot\mathbf{x}'')^2}.$$

A curve with non-vanishing curvature is planar if and only if its torsion vanishes identically. If the curve is planar, then $\mathbf{b}$ is constant, $\dot{\mathbf{b}}=0$, and $\tau=0$. Conversely, if $\tau=0$, then $\mathbf{b}=\mathbf{c}$, and integrating $\mathbf{b}\cdot\mathbf{t}=0$ gets us $\mathbf{b}\cdot\mathbf{x}(s)=c'$; that is, the curve represented by $\mathbf{x}$ lies in a plane orthogonal to the constant vector $\mathbf{b}$.

15. Formulae of Frenet.

Since the triplet of vectors $\mathbf{t}$, $\mathbf{p}$, and $\mathbf{b}$ form an orthogonal basis, it must be possible to represent the first derivatives $\dot{\mathbf{t}}$, $\dot{\mathbf{p}}$ and $\dot{\mathbf{b}}$ of each vector as a linear combination of the aforementioned triple; the corresponding formulae are called the formulae of Frenet.

We have already investigated two of the three formulae of Frenet,

$$\dot{\mathbf{t}} = \kappa\mathbf{p}, \qquad \dot{\mathbf{b}} = -\tau\mathbf{p},$$

and shall now find an analogous representation of the vector $\dot{\mathbf{p}}$. Differentiating $\mathbf{p}\cdot\mathbf{p} = 1$ gives $\mathbf{p}\cdot\dot{\mathbf{p}} = 0$, so $\dot{\mathbf{p}}$ is orthogonal to $\mathbf{p}$ and we can write $\dot{\mathbf{p}} = a\mathbf{t} + c\mathbf{b}$. Scalar multiplication by $\mathbf{t}$ and $\mathbf{b}$ respectively yields

$$a = \dot{\mathbf{p}}\cdot\mathbf{t} = -\mathbf{p}\cdot\dot{\mathbf{t}} = -\kappa\mathbf{p}\cdot\mathbf{p} = -\kappa$$
$$c = \dot{\mathbf{p}}\cdot\mathbf{b} = -\mathbf{p}\cdot\dot{\mathbf{b}} = \tau$$

where the second equality in each case is a general result between orthogonal vectors:

$$\mathbf{a}\cdot\mathbf{b} = 0 \quad\Longleftrightarrow\quad 0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\dot{\mathbf{b}} + \mathbf{b}\cdot\dot{\mathbf{a}} = 0.$$

We have thus obtained all three formulae of Frenet; written in matrix notation,

$$\begin{bmatrix}\dot{\mathbf{t}}\\ \dot{\mathbf{p}}\\ \dot{\mathbf{b}}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{p}\\ \mathbf{b}\end{bmatrix}.$$

Note that the coefficient matrix is skew-symmetric. Indeed, the formula of Frenet for $\dot{\mathbf{p}}$ can be derived by first showing that the coefficient matrix must be skew-symmetric,

$$c_{ik} = \left(\sum_j c_{ij}\mathbf{a}_j\right)\cdot\mathbf{a}_k = \dot{\mathbf{a}}_i\cdot\mathbf{a}_k = -\mathbf{a}_i\cdot\dot{\mathbf{a}}_k = -\mathbf{a}_i\cdot\left(\sum_j c_{kj}\mathbf{a}_j\right) = -c_{ki},$$

and hence using the known formulae of Frenet for $\dot{\mathbf{t}}$ and $\dot{\mathbf{b}}$.

16. Motion of the trihedron, vector of Darboux.

Because the vectors $\mathbf{t}$, $\mathbf{p}$, and $\mathbf{b}$ always have the same mutual position and constant unit length, we may imbed these vectors in a rigid body K which performs the same motion as the trihedron, and consider our problem as one involving rigid bodies. We shall consider here only the rotation—and not translation—of the trihedron along increasing values of the arclength parameter s.

Let P be any point of a rotating body K, and denote by $\mathbf{r}$ the position vector of P referred to a coordinate system with origin on the axis of rotation. $\mathbf{r}$ is of the form $\mathbf{r} = u\mathbf{t}+v\mathbf{p}+w\mathbf{b}$; assuming that the point moving along the curve C under consideration has the constant velocity 1 we may equate the arclength s of C with the time t, and the velocity vector of P is thus of the form $\mathbf{v} = \dot{\mathbf{r}} = u\dot{\mathbf{t}} + v\dot{\mathbf{p}} + w\dot{\mathbf{b}}$. The rotation (angular velocity) vector $\mathbf{d}$ is given by $\mathbf{v} = \mathbf{d}\times\mathbf{r}$. $\mathbf{d}$ is hence orthogonal to $\mathbf{v}$, which, by the formulae of Frenet, lies in the plane spanned by $\dot{\mathbf{t}}$ and $\dot{\mathbf{p}}$. Hence $\mathbf{d}$ has the direction of the vector

$$\dot{\mathbf{t}}\times\dot{\mathbf{p}} = \kappa\mathbf{p}\times(-\kappa\mathbf{t}+\tau\mathbf{b}) = \kappa^2(\mathbf{t}\times\mathbf{p}) + \kappa\tau(\mathbf{p}\times\mathbf{b}) = \kappa(\kappa\mathbf{b}+\tau\mathbf{t}),$$

and can be represented in the form $\mathrm{d} = c(s)(\tau\mathbf{t}+\kappa\mathbf{b})$. The magnitude of $\mathbf{d}$ is a geometric property intrinsic to the curve, and hence cannot depend on u, v, or w. We can thus determine $c(s)$ by a shrewd choice of u, v, and w; set $v = w = 0$, as a result of which $\mathbf{v}\to\dot{\mathbf{t}}$ and $\mathbf{r}\to\mathbf{t}$. We have

$$\mathbf{v} = \mathbf{d}\times\mathbf{r} \quad\Longleftrightarrow\quad \dot{\mathbf{t}} = c(\tau\mathbf{t}+\kappa\mathbf{b})\times\mathbf{t} = c\kappa\mathbf{p} = \kappa\mathbf{p}$$

by a formula of Frenet, and we find that $c(s)\equiv 1$. Hence the rotation vector $\mathbf{d}$ is given by $\mathbf{d} = \tau\mathbf{t}+\kappa\mathbf{b}$, and by choosing separately $u = w = 0$ and $u = v = 0$, we see that we can rewrite the formulae of Frenet in the form

$$\dot{\mathbf{t}} = \mathbf{d}\times\mathbf{t}, \quad \dot{\mathbf{p}} = \mathbf{d}\times\mathbf{p}, \quad \dot{\mathbf{b}} = \mathbf{d}\times\mathbf{b}.$$