Problem. A block is placed on a plane inclined at an angle $\theta$. The coefficient of friction between the block and the plane is $\mu=\tan \theta$. The block initially moves horizontally along the plane at a speed $V$. In the long-time limit, what is the speed of the block?

Solution 1. (Constant of the motion.) Since $F_{f}=\mu N=\tan \theta(m g \cos \theta)=m g \sin \theta$, the magnitudes of the gravitational and friction forces acting on the block are equal. We would like to claim that

$$
a_{y}=-a_{t} .
$$

( $t$ stands for tangent; +y is taken in the direction down the plane; the negative sign is because friction acts oppositely the tangential direction.) Because the two forces are not orthogonal, however, we must thus take both forces into account when considering $a_{y}$ and $a_{t}$. We can show by explicit evaluation or by symmetry arguments that the magnitude of the net acceleration in either direction is indeed the same, so our claim is proven. Direct integration then yields

$$
v_{y}=-v_{t}+C=-v+V,
$$

where the constant $C$ is obtained from our initial conditions. (The velocity is always in the tangential direction, so $v_{t}=v$.) The constant $C$ is a constant of the motion, because it is invariant over the entire trajectory. In the long-time limit, by inspection, the motion is down the plane, so $v=v_{y}$ and $v_{y}=V / 2$.

Remark. One might be suspicious about the integration: whereas $\hat{\mathbf{y}}$ is a constant vector, $\hat{\mathbf{t}}$ changes with the direction of motion, and it appears that we may need to take that into consideration when integrating $a_{t}$. It turns out that we do not: consider

$$
\mathbf{v}=v \hat{\mathbf{t}} \quad \text { and } \quad \mathbf{a}=\dot{\mathbf{v}}=\dot{v} \hat{\mathbf{t}}+v \dot{\hat{\mathbf{t}}}=\dot{v} \hat{\mathbf{t}}+v \dot{\varphi} \hat{\mathbf{n}}=: a_{t} \hat{\mathbf{t}}+a_{n} \hat{\mathbf{n}} .
$$

(The unit tangent vector $\hat{\mathbf{t}}$ is of unit norm, and its time derivative must therefore be orthogonal to it. This, along with the constraint that $\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{t}} \times \hat{\mathbf{n}}$ form a right-handed coordinate system, defines the unit normal vector $\hat{\mathbf{n}} . \varphi$ is the angle the tangent vector to the curve makes with the horizontal. The argument for

$$
\dot{\hat{\mathbf{t}}}=\dot{\varphi} \hat{\mathbf{n}} \quad \text { is equivalent to that for } \quad \dot{\hat{\mathbf{r}}}=\dot{\theta} \hat{\boldsymbol{\theta}} .)
$$

Thus we see that $\dot{v}=a_{t}$, and our direct integration is acceptable.
Remark. Intuitively, why should the tangential deceleration from friction impart the same change in speed as does the vertical acceleration from gravity, the trajectory of the block being curved?

The derivation above shows that the crux to this puzzle is that, in both cases, the direction of the acceleration is also the direction of the velocity. In this case the change in speed is linear in time, whereas in the general case we would have to resolve the velocity into orthogonal components and use the Pythagorean theorem. The frictional force imparts a constant deceleration (in $\hat{\mathbf{t}}$ ) over the same period of time as does the gravitational force (in $\hat{\mathbf{y}}$ ), so it is reasonable that both would lead to the same change in speed in $v_{t}=v$ and $v_{y}$ respectively.
(Continued on back.)

Solution 2. (Newton's equations.) We will solve for the trajectory directly. The preferred coordinates here are the intrinsic coordinates $s$ and $\varphi$, respectively the arclength and angle the tangent to the curve makes with the horizontal. These coordinates are natural here because the frictional force acts opposite the tangential direction, and the tangential direction is defined with respect to the curve (that is, intrinsically). Referring to Solution 1 and recalling $\dot{s}=v$, the decomposition of the acceleration in the tangential and normal directions yields

$$
\begin{aligned}
m \ddot{s} & =m g \sin \theta(\sin \varphi-1) \\
m \dot{s} \dot{\varphi} & =m g \sin \theta \cos \varphi
\end{aligned}
$$

For convenience, substitute $u=\dot{s} / g \sin \theta$ and eliminate $u$, obtaining

$$
\begin{aligned}
\dot{u} & =\sin \varphi-1 \\
u \dot{\varphi} & =\cos \varphi
\end{aligned}
$$

Eliminating $u$, some algebra leads to the directly integrable differential equation

$$
\frac{\ddot{\varphi}}{\dot{\varphi}}=(\sec \varphi-2 \tan \varphi) \dot{\varphi} \quad \Longleftrightarrow \quad \dot{\varphi}=A \cos \varphi(1+\sin \varphi)
$$

From $u \dot{\varphi}=\cos \varphi$, we get

$$
\dot{\varphi(0)}=\frac{\cos \varphi(0)}{u(0)}=\frac{1}{u(0)}=\frac{g \sin \theta}{\dot{s}(0)}=\frac{g \sin \theta}{V} \quad \Longleftrightarrow \quad A=\frac{g \sin \theta}{V}
$$

and, solving for $\dot{s}$, we have

$$
\dot{s}=g \sin \theta u=\frac{g \sin \theta \cos \varphi}{\dot{\varphi}}=\frac{V}{1+\sin \varphi} \quad \text { and } \quad \lim _{t \rightarrow \infty} \dot{s}=\lim _{\varphi \rightarrow \pi / 2} \dot{s}=\frac{V}{2}
$$

