Math 98 Decal: Conceptual Mathematics

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Spring 2019

Contents

1 Course Description

During the course, students will learn basic categorical concepts and how to think structurally rather than internally (in the sense of Lawvere). They will be able to apply categorical concepts to other mathematical fields, and perhaps even outside the mathematical world.

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2 Required Materials

- Course notes online at the course webpage
- *Conceptual Mathematics* by Lawvere and Schanuel (you can obtain an electronic copy by going to Oskicat and clicking on the links either on campus or via proxy).
- *Introduction to Categories and Categorical Logic* by Abramsky and Tzevelekos. The text is available here.

3 Possibly Helpful Materials

3.1 Motivation

Some of these might be helpful supplemental reading especially at the beginning of the course.

Structuralism and Nominalism section of this page

`http://philomatica.org/wp-content/uploads/2012/12/id_talk.pdf`

What Numbers Could Not Be by Paul Benaceraf.

3.2 General Reference

The following either supplement what we will cover or give a more detailed treatment of what we will cover. This list **is in no way exhaustive! Feel free to find some other texts if you like.** These books usually assume an undergraduate background in math. The ones in 3.3 don't.

Categories for the Working Mathematician (either edition) by Saunders Mac Lane. (Classic reference for category theory)

Abstract and Concrete Categories: The Joy of Cats by Jiří Adámek, Horst Herrlich, and George E. Strecker. (Lots of examples, beware of older terminology e.g. their notion of quasicategory is not a model for $(\infty, 1)$-categories!)

Basic Category Theory by Leinster. (Nice introductory textbook)

Category Theory in Context by Emily Riehl. (Gives lots of examples)

3.3 Other Introductions

Category Theory for the Sciences by David Spivak

Category Theory for Programmmers by Bartosz Milewski

Category Theory by Steve Awodey

4 Prerequisites/Corequisites

4.1 If you haven't seen category theory before:

It would be great if you had some idea of what a vector space and a group is (*intuitively*) but this isn't necessary. We will go over such structures around weeks 5 and 6. The bare minimum should be some familiarity with matrices and mathematical induction.

4.2 If you have:

We're hoping to cover some more specialized topics that aren't covered in, say, Mac Lane albeit less in-depth. If you'd like, you can spend more time looking at something from 5.4. We can try to work something out if you still want the units. Category theory is a *vast* subject, and there's a lot that we don't know ourselves.

5 Course Structure

5.1 Class Structure

This class will be taught similar to a discussion section. Each week we will meet once for an one and a half hours to talk about readings. To help facilitate discussion, each week you will be asked to send in a question about the reading.

We want to make this class fun and engaging for people whose backgrounds might be from outside of math. Our backgrounds lean towards algebra (but we're not experts in any way!), so we might learn as much from you as we hope you learn from us.

5.2 Weekly Questions

Each week students should come up with at least one question from the reading. You can either turn the question in on paper during class or email it to us any time before class (e.g. the day before). If you can, please email in the questions the day before so that we can properly prepare an answer. The questions don't need to be involved as long as they're questions about the reading.

Sample Question: In a monoidal category, why do we only ask that $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ be an isomorphism satisfying coherence conditions rather than just asking for $(a \otimes b) \otimes c = a \otimes (b \otimes c)$?

If you don't have a question, if there is an exercise in the reading you can turn in a solution to an exercise in place of a question. *Attendance is encouraged; please email us if for some reason you cannot attend.*

5.3 Report

There will be a report which will be due on the last week of instruction (the week before RRR week).

Rather than examining you on the material, we would like you to write about a related subject which you are interested in. The final assignment should be written as if you were lecturing on a topic, and can either be expository or include some original material if you wish.

It might be helpful to briefly glance at the readings since they are available before hand and think a little bit about what you wish to write about as we move along the semester. For finding ideas, it might also be helpful to glance at the table of contents, prefaces, and abstracts (if applicable) for the titles listed in 5.4.

As a matter of logistics, please let us know what you want to do when you've decided on a topic or if you've decided to change topics (it's perfectly okay if you find that you didn't like topic you chose initially). You can let us know in person or via email.

We look forward to reading your works!

5.4 Suggestions for the Report

We will put up a PDF with some suggestions before the halfway point through the semester. It will appear in the `Schedule` section of the webpage.

Note: These books assume a varying degree of familiarity with category theory!

An Invitation to General Algebra and Universal Constructions (This was the textbook used for general algebra with Professor Bergman; it has plenty of examples)

Categories for the Working Philosopher

Category Seminar

Categories in Continuum Mechanics

Categories in Algebra, Geometry, and Mathematical Physics

Introduction to Higher Order Categorical Logic

Malcev, Protomodular, Homological, and Semi-Abelian Categories

Category Theory for Scientists

Higher Operads, Higher Categories

Topoi: The Categorical Analysis of Logic

Sheaves in Geometry and Logic

Category Theory for Computing Science

New Structures for Physics

... and lots more!

We've listed books, but there are other resources as well. For example: nLab, nCafé, people's webpages (John Baez, Steve Awodey, Emily Riehl, Michael Shulman,...), `kerodon.net`, or The Catsters.

5.5 Grading Policy

- 40% reading questions (graded primarily on completion)
- 60% final assignment

The class is graded on a pass/no pass basis. Students should get at least 70% to pass the course.

5.6 Accomodations and Emergencies

If you need an accomodation, please let us know as soon as possible.

In case something important or urgent comes up, please contact us as soon as possible and we will try to work something out.

6 Schedule and Readings

The schedule is tentative and subject to change. Since our class time is limited, please at least skim the readings before discussion and bring a question to class. CM means *Conceptual Mathematics*, and ICCL means *Introduction to Categories and Categorical Logic* by Abramsky and Tzevelekos.

Week 04, 02/11 - 02/15: Introduction

- We'll talk about our backgrounds and expectations
- We'll present some shortcomings to analytic approaches for (as an example) defining the natural numbers. We'll introduce structural approaches instead (something similar to ETCS, don't get to talk about what lives inside of them; take functions as *primitives*). Talk about the special place that natural numbers takes inside this categories.
- You don't need to bring a question to the first class, but maybe think about what you would like to learn and are prepared to put in
- (After our first meeting) Article 1 of CM [1]

Week 05, 02/18 - 02/22: Categories, **Set**, Isomorphisms

- We'll talk about the problem of equality. Bijection of sets vs equality of sets. We'll also bring up certain types of arrows: epimorphisms, monomorphisms, restractions, sections, and they relate via the dual of a category. We'll also briefly look at some examples of universal properties as part of the discussion.
- Pages 1-11 of *When is one thing equal to some other thing?* by Mazur
- Article 2 of CM
- *(Optional)* Briefly gloss over `http://philomatica.org/wp-content/uploads/2012/12/id_talk.pdf`

[1] if you're confused about something, the sessions are like discussions, please first take a look at the sections corresponding to the topic in the article; if you still have questions then you can submit it

Week 06, 02/25 - 03/01: (Directed) Graphs, Functors, Natural Transformations

- We'll look at examples of categories, functors (graphs, actions), natural transformations (directed homotopy). We'll have a more in-depth look at structures like monoids, groups, topological spaces, and actions.
- Section 1.1 of ICCL
- *Mathematical Physics* by Robert Geroch gives plenty of examples of structures we care about and does some book keeping using some categorical concepts; the book doesn't have much actual physics in it, keeping true to the *mathematical* part of mathematical physics

Week 07, 03/04 - 03/08: Continuation of week 6

- Continuation of our examples and explanations. We'll have our first look at algebraic invariants, primarily focusing on Π_1 and Brouwer's fixed point theorem. We'll also look at the basic idea of a (single sorted, finitary) Lawvere theory and a monad. The machinery to make the connection explicit will have to wait.
- Session 10

Week 08, 03/11 - 03/15: Examples, Space

- We'll talk about the idea of presheaves and why we might think of them as generalized space, using the idea that we are probing a space by some category of nice spaces or local models. Examples. Yoneda part 1 (the Yoneda embedding is fully faithful).
- Sections 1-6 of Article 3
- Session 13

Week 09, 03/18 - 03/22: More examples, Quantity

- Talk about the idea of copresheaves and why we might think of them as generalized quantities. Examples.
- Sections 7-12 of Article 3

Week 10, 03/25 - 03/29: ***Spring Break***

Week 11, 04/01 - 04/05: Continuation of weeks 5 and 6

- Isbell duality part 1. Examples.

Week 12, 04/08 - 04/12: Universal Constructions

- Talk about universal mapping properties, give some examples such as free monoid, category generated by a graph; terminal, initial, zero objects, examples such as empty set, 1-point set in well-pointed categories, $\mathbb{N}$ in the category of sets, endomorpisms, and a distinguished element and $\mathbb{Z}$ in **Ring**.
- Article 4

Week 13, 04/15 - 04/19: Continuation of week 8

- Limits and colimits (product, coproduct, pullback, pushout)
- Section 1.5.1 of ICCL

Week 14, 04/22 - 04/26: Adjoint Functors, Yoneda Lemma part 2

- Free forgetful, limits and colimits in terms of adjoint, hom definition vs universal mapping property definitions of adjoint functors. Isbell duality part 2.
- Section 1.5.2 of ICCL

Week 15, 04/29 - 05/03: Continuation of week 11

- More on the Yoneda lemma, co-Yoneda lemma, more examples of adjunctions.
- *The Yoneda Lemma: What's it all about?* by Leinster
- ***Report Due***