Design-Adaptive Nonparametric Estimation of Conditional Quantile Derivatives

SUPPLEMENTARY MATERIAL

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This supplement contains detailed proofs of Proposition 3.1 and Theorems 3.2 and 3.3 as given in the paper in question. For the sake of completeness, the appropriate regularity conditions are reproduced in Section 1 below. The proof of Proposition 3.1 appears in Section 2, while those of Theorems 3.2 and 3.3 appear in Section 3.

1 Regularity conditions

Assumption A.1 (Conditional distributions) The conditional distribution of Y given $X = x$ is absolutely continuous with respect to Lebesgue measure for each $x \in \mathcal{X}$. The corresponding density function $f_{Y|X=x}(\cdot)$ satisfies

(1) $0 < \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y|X=x}(q(\alpha, x)) < \infty$,

(2) while the first derivative of $f_{Y|X=x}(\cdot)$ satisfies $\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f^{(1)}_{Y|X=x}(q(\alpha, x)) \right| < \infty$.

Assumption A.2 (Conditional quantile functions)

(1) $\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} |D^{\nu}_x q(\alpha, x)| < \infty$.

(2) $\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| D^{\nu}_x q^{(2)}(\alpha, x) \right| < \infty$.

(3) All partial derivatives of $q(\alpha, x)$ with respect to x exist to at least sixth order for each $x \in \mathcal{X}$; these derivatives are all uniformly bounded over $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$.

Assumption A.3 (Design)

(1) $g(x) > 0$ for all $x \in \mathcal{X}$, and $\sup_{x \in \mathcal{X}} g(x)$, $\sup_{x \in \mathcal{X}} \left| g^{(1)}(x) \right|$ and $\sup_{x \in \mathcal{X}} \left| g^{(2)}(x) \right|$ are all finite.

(2) $\frac{d^{L-1}}{dx^{L-1}} g(x) < \infty$, where $L \geq 2$ is as given below in Assumption 1.

(3) $\sup_{x \in \mathcal{X}} |D^{\nu}_x g(x)| < \infty$.

(4) $\sup_{x \in \mathcal{X}} \left| D^{\nu}_x g^{(2)}(x) \right| < \infty$.

Assumption A.4 (Kernel functions)

(1) $\kappa : \mathbb{R} \to \mathbb{R}$ is a symmetric density function with finite moments to at least second order. In addition, $\int u^2 \kappa^2(u) du < \infty$.

(2) $k : \mathbb{R} \to \mathbb{R}$ satisfies

a) $\int k(u) du = 1$.

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b) For some integer $L \geq 2$, $\int u^{\alpha}k(u)du = 0$ for all $\alpha \in \mathbb{Z}_+$ with $1 \leq \alpha \leq L-1$ and $\int u^{L}k(u)du \in (0,\infty)$.

c) $\int k^4(u)du < \infty$.

d) k is Lipschitz, with $|k(u+v) - k(u)| < c_k(u)|v|$ for some finite real-valued function $c_k(\cdot)$ on $\mathbb{R}$.

e) Both $\int u^{16}k^2(u)du$ and $\int u^{28}k^4(u)du$ are finite.

(3) $K : \mathbb{R} \to \mathbb{R}$ satisfies

a) $K(\cdot)$ is a symmetric density function supported on $\mathbb{R}$ with finite moments to at least sixth order.

b) $\int \left(\frac{d^{\nu}}{du^{\nu}}K(u)\right)^4 du < \infty$.

c) $\left|\int u \left(\frac{d^{\nu}}{du^{\nu}}K(u)\right)^2 du\right|$ and $\left|\int u \left(\frac{d^{\nu}}{du^{\nu}}K(u)\right)^4 du\right|$ are both finite.

Assumption A.5 (Trimming and smoothing)

(1) For some constant $\zeta > 0$, the trimming function $\tau_n(u)$ satisfies

$$\tau_n(u) = \begin{cases} 1 & , \quad u \geq 2n^{-\zeta} \\ 0 & , \quad u \leq n^{-\zeta} \\ \bar{\tau}_n(u) & , \quad u \in \left(n^{-\zeta}, 2n^{-\zeta}\right), \end{cases}$$

where $\bar{\tau}_n(u)$ is a distribution function with the form $\bar{\tau}_n(u) = \int_{-\infty}^{u} n^{\zeta}\sigma\left(n^{\zeta}t - 1\right)dt$, where $\sigma(\cdot)$ is a differentiable density function uniformly bounded and supported on $[0,1]$ with $\sigma(0) = \sigma(1) = 0$ and $0 < \left|\sigma^{(1)}(0)\right|, \left|\sigma^{(1)}(1)\right| < \infty$.

(2) $h_g \geq 0$ satisfies the following:

a) $h_g \to 0$ and $nh_g \to \infty$ as $n \to \infty$;

b) $n^{\zeta}\left(h_g^2 + 1/\sqrt{nh_g}\right) \to 0$ as $n \to \infty$ where ζ is the trimming constant given above in part (1).

(3) $h_q \geq 0$ satisfies the following:

a) $h_q \to 0$, $nh_q \to \infty$;

b) $\sqrt{nh_q} \cdot h_q^{L-1} \to 0$ as $n \to \infty$ for $L \geq 2$ as given above in Assumption 1.

(4) $h \geq 0$ satisfies the following:

a) $h \to 0$, $nh^{1+2\nu} \to \infty$;

b) $\sqrt{nh^{1+2\nu}} \cdot h \to 0$ and $h^{1+2\nu}/h_q \to 0$ as $n \to \infty$.

2 Proof of Proposition 3.1

2.1 Preliminaries

The proof of Proposition 3.1 relies on a number of preliminary definitions, which for the sake of completeness are recapitulated here.

For each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$, let

$$q_g^{(1)}(\alpha, x) \equiv \frac{d}{dx}\left[\frac{q(\alpha, x)}{g(x)}\right]$$

and

$$q_g^{(2)}(\alpha, x) \equiv \frac{d^2}{dx^2}\left[\frac{q(\alpha, x)}{g(x)}\right].$$

Let $\rho_\alpha(u) \equiv u(\alpha - 1(u<0))$, and let $Y_i^* \equiv Y_i/\hat{g}_n(X_i)$ for each $i = 1,\ldots,n$. Note that for $\boldsymbol{\delta} \equiv (\ \delta_0, \quad \delta_1\ )^\top$, the function

$$\begin{aligned} Z_n(\boldsymbol{\delta}) \quad &\equiv \quad \sum_{i=1}^{n}\left(\rho_\alpha\left(Y_i^* - \frac{q(\alpha,x)}{g(x)} - q_g^{(1)}(\alpha,x)\cdot(X_i-x) - \frac{1}{\sqrt{nh_q}}\left[\delta_0 + \delta_1 \cdot h_q^{-1}(X_i - x)\right]\right)\right. \\ &\left. -\rho_\alpha\left(Y_i^* - \frac{q(\alpha,x)}{g(x)} - q_g^{(1)}(\alpha,x)\cdot(X_i-x)\right)\right) k\left(h_q^{-1}(X_i-x)\right)\tau_n\left(\hat{g}_n(X_i)\right) \end{aligned}$$

is convex and minimised for each $(\alpha,x) \in \mathcal{A}\times\mathcal{X}$ at

$$\begin{bmatrix} \hat{\delta}_{n0}(\alpha,x) \\ \hat{\delta}_{n1}(\alpha,x) \end{bmatrix} = \sqrt{nh_q}\begin{bmatrix} \hat{q}_n^*(\alpha,x) - \frac{q(\alpha,x)}{g(x)} \\ h_q\left(\hat{q}_{n1}^*(\alpha,x) - q_g^{(1)}(\alpha,x)\right) \end{bmatrix}.$$

The conclusion of Proposition 3.1, namely that $\sqrt{nh_q}\,(\hat{q}_n^*(\alpha,x) - q(\alpha,x)/g(x)) = O_p(1)$ for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$, is deduced by first showing the convergence in distribution of the vector $(\ \hat{\delta}_{n0}(\alpha,x), \quad \hat{\delta}_{n1}(\alpha,x)\ )^\top$ for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$.

The convergence in distribution of $(\ \hat{\delta}_{n0}(\alpha,x), \quad \hat{\delta}_{n1}(\alpha,x)\ )^\top$ is deduced from the asymptotic behaviour of the criterion function $Z_n(\boldsymbol{\delta})$ for each $\boldsymbol{\delta}\in\mathbb{R}^2$. From "Knight's identity" (Knight 1998; Koenker 2005, p. 121), we have

$$Z_n(\boldsymbol{\delta}) = Z_{n1}(\boldsymbol{\delta}) + Z_{n2}(\boldsymbol{\delta}), \tag{1}$$

where for

$$\psi_\alpha(u) \equiv \alpha - 1(u<0),$$

we have

$$\begin{aligned} Z_{n1}(\boldsymbol{\delta}) \quad &= \quad -\frac{1}{\sqrt{nh_q}}\sum_{i=1}^{n}\left[\delta_0 + h_q^{-1}(X_i-x)\,\delta_1\right]\psi_\alpha\left(u_i^*(x)\right)k\left(h_q^{-1}(X_i-x)\right)\tau_n\left(\hat{g}_n(X_i)\right); \\ Z_{n2}(\boldsymbol{\delta}) \quad &= \quad \sum_{i=1}^{n}\int_0^{v_{ni}(x)}\left(1\left(u_i^*(x)\le s\right) - 1\left(u_i^*(x)\le 0\right)\right)ds, \end{aligned}$$

and where

$$\begin{aligned} u_i^*(x) \quad &\equiv \quad Y_i^* - \frac{q(\alpha,x)}{g(x)} - (X_i-x)\cdot q_g^{(1)}(\alpha,x); \\ v_{ni}(x) \quad &\equiv \quad \frac{1}{\sqrt{nh_q}}\left[\delta_0 + h_q^{-1}(X_i-x)\,\delta_1\right]k\left(h_q^{-1}(X_i-x)\right)\tau_n\left(\hat{g}_n(X_i)\right). \end{aligned}$$

Begin by examining the large-sample behaviour of $Z_{n2}(\boldsymbol{\delta})$ on $\mathcal{A}\times\mathcal{X}$ before proceeding to an analysis of $Z_{n1}(\boldsymbol{\delta})$. To start, recall that

$$\hat{g}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\kappa_{h_g}(X_i-x),$$

where $\kappa_{h_g}(t) \equiv h_g^{-1}\kappa\left(h_g^{-1}t\right)$, and where $\kappa:\mathbb{R}\to\mathbb{R}$ is a kernel function satisfying the conditions of part (1) of Assumption A.4, and h_g is a bandwidth satisfying $h_g\to 0$ with $nh_g\to\infty$. Note that

$$\begin{aligned} & E\left[\kappa_{h_g}(X_1-x)\right] \\ = \quad & \int \kappa_{h_g}(x_1-x)\,g(x_1)\,dx_1 \\ = \quad & \int h_g^{-1}\kappa\left(h_g^{-1}(x_1-x)\right)g(x_1)\,dx_1 \\ = \quad & \int \kappa(t_1)\,g(x+h_g t_1)\,dt_1 \\ = \quad & g(x)\int\kappa(t_1)\,dt_1 + h_g g^{(1)}(x)\int t_1\kappa(t_1)\,dt_1 \\ & +\frac{h_g^2}{2}\int t_1^2 g^{(2)}\left(\xi(x,t_1)\right)\kappa(t_1)\,dt_1, \end{aligned}$$

where $\xi(x, t_1)$ is a point on the line segment between x and $x + h_g t_1$.

It follows that

$$\sup_{x\in\mathcal{X}} \left| E\left[\kappa_{h_g}(X_1 - x)\right] - g(x)\right| = O\left(h_g^2\right), \tag{2}$$

where the various conditions on $\kappa(\cdot)$ and the assumption that $\sup_{x\in\mathcal{X}} \left|g^{(2)}(x)\right| < \infty$ have been invoked.

Note also that

$$\begin{aligned} & E\left[\hat{g}_n^2(x)\right] \\ = \;& E\left[\frac{1}{n^2}\sum_{i,j}\kappa_{h_g}(X_i - x)\,\kappa_{h_g}(X_j - x)\right] \\ = \;& \frac{1}{n}E\left[\kappa_{h_g}^2(X_1 - x)\right] + \frac{n(n-1)}{n^2}\left(E\left[\kappa_{h_g}(X_1 - x)\right]\right)^2. \end{aligned}$$

We have

$$\begin{aligned} & E\left[\kappa_{h_g}^2(X_1 - x)\right] \\ = \;& \int \kappa_{h_g}^2(x_1 - x)\, g(x_1)\, dx_1 \\ = \;& \int h_g^{-2}\kappa^2\left(h_g^{-1}(x_1 - x)\right) g(x_1)\, dx_1 \\ = \;& h_g^{-1}\int \kappa^2(t_1)\, g(x + h_g t_1)\, dt_1 \\ = \;& h_g^{-1}\left(g(x)\int \kappa^2(t_1)\, dt_1 + h_g g^{(1)}(x)\int t_1\kappa^2(t_1)\, dt_1 \right. \\ & \left. + \frac{h_g^2}{2}\int t_1^2 g^{(2)}(\xi(x, t_1))\,\kappa^2(t_1)\, dt_1\right). \end{aligned}$$

As such,

$$\begin{aligned} & E\left[\hat{g}_n^2(x)\right] \\ = \;& \frac{1}{nh_g}g(x)\int \kappa^2(t_1)\, dt_1 + O\left(\frac{h_g^2}{nh_g}\right) \\ & + g^2(x) + h_g^2 g(x)\int g^{(2)}(\xi(x, t_1))\, t_1^2\kappa(t_1)\, dt_1 \\ & + O\left(h_g^4\right). \end{aligned}$$

Deduce that

$$\begin{aligned} & Var\left[\hat{g}_n(x)\right] \\ = \;& E\left[\left(\hat{g}_n(x) - E\left[\hat{g}_n(x)\right]\right)^2\right] \\ = \;& E\left[\left(\hat{g}_n(x) - E\left[\kappa_{h_g}(X_1 - x)\right]\right)^2\right] \\ = \;& E\left[\hat{g}_n^2(x)\right] - \left(E\left[\kappa_{h_g}(X_1 - x)\right]\right)^2 \\ = \;& g^2(x) + \frac{1}{nh_g}g(x)\int \kappa^2(t_1)\, dt_1 + h_g^2 g(\boldsymbol{x})\int g^{(2)}(\xi(x, t_1))\, t_1^2\kappa(t_1)\, dt_1 \\ & - g^2(x) - h_g^2 g(x)\int g^{(2)}(\xi(x, t_1))\, t_1^2\kappa(t_1)\, dt_1 + O\left(h_g^4\right) \\ = \;& O\left(\frac{1}{nh_g} + h_g^4\right). \end{aligned}$$

It follows that

$$\sup_{x\in\mathcal{X}}\left|\hat{g}_n(x) - E\left[\kappa_{h_g}(X_1 - x)\right]\right| = O_p\left(\frac{1}{\sqrt{nh_g}} + h_g^2\right), \tag{3}$$

where the assumptions that $\sup_{x\in\mathcal{X}} g(x) < \infty$ and $\sup_{x\in\mathcal{X}} \left|g^{(2)}(x)\right| < \infty$ have been invoked.

Combine (2) and (3) to deduce that

$$\sup_{x\in\mathcal{X}} |\hat{g}_n(x) - g(x)| = O_p\left(h_g^2 + \frac{1}{\sqrt{nh_g}}\right). \tag{4}$$

This deduction also relies on the assumptions that $\sup_{x\in\mathcal{X}} g(x) < \infty$, $\sup_{x\in\mathcal{X}} \left|g^{(2)}(x)\right| < \infty$ and $\int t_1^2 \kappa^2 (t_1)\, dt_1 < \infty$.

Now consider that

$$\tau_n(\hat{g}_n(x)) - \tau_n(g(x)) = (\hat{g}_n(x) - g(x))\, \tau_n^{(1)}(g^*(x)),$$

where $g^*(x)$ is an intermediate point. We have

$$\begin{aligned}
& \sup_{x\in\mathcal{X}} |\tau_n(\hat{g}_n(x)) - \tau_n(g(x))| \\
\leq\ & \sup_{x\in\mathcal{X}} |\hat{g}_n(x) - g(x)| \cdot \sup_{u\in\left(n^{-\zeta}, 2n^{-\zeta}\right)} n^{\zeta}\sigma\left(n^{\zeta}u - 1\right) \\
=\ & O_p\left(n^{\zeta}\left(h_g^2 + \frac{1}{\sqrt{nh_g}}\right)\right)
\end{aligned} \tag{5}$$

by virtue of (4) and the uniform boundedness of $\sigma(\cdot)$ on $[0, 1]$.

Returning to $Z_{n2}(\boldsymbol{\delta})$, define

$$\bar{Z}_{n2}(\boldsymbol{\delta}) \equiv \sum_{i=1}^{n} \int_0^{\bar{v}_{ni}(x)} \left(1\left(\bar{u}_i^*(x) \leq s\right) - 1\left(\bar{u}_i^*(x) \leq 0\right)\right) ds,$$

where

$$\begin{aligned}
\bar{u}_i^*(x) &= \frac{Y_i}{g(X_i)} - \frac{q(\alpha, x)}{g(x)} - (X_i - x)\, q_g^{(1)}(\alpha, x), \\
\bar{v}_{ni}(x) &= \frac{1}{\sqrt{nh_q}} \left[\delta_0 + h_q^{-1}(X_i - x)\,\delta_1\right] k\left(h_q^{-1}(X_i - x)\right) \tau_n(g(X_i)).
\end{aligned} \tag{6}$$

Write

$$Z_{n2}(\boldsymbol{\delta}) = \sum_{i=1}^{n} E\left[Z_{n2i}(\boldsymbol{\delta})\middle| X_i\right] + \sum_{i=1}^{n} \left(Z_{n2i}(\boldsymbol{\delta}) - E\left[Z_{n2i}(\boldsymbol{\delta})\middle| X_i\right]\right),$$

and

$$\bar{Z}_{n2}(\boldsymbol{\delta}) \equiv \sum_{i=1}^{n} \int_0^{\bar{v}_{ni}(x)} \left(1\left(\bar{u}_i^*(x) \leq s\right) - 1\left(\bar{u}_i^*(x) \leq 0\right)\right) ds.$$

Let $F_i(\cdot)$ denote the conditional distribution function $F_{Y_i|X_i}(\cdot)$ and $f_i(\cdot)$ denote the corresponding conditional density function $f_{Y_i|X_i}(\cdot)$. We note that for $T_i \equiv h_q^{-1}(X_i - x)$ $(i \in \{1, \ldots, n\})$ and ξ_i a point on the line segment between x and $X_i = x + H_q T_i$ that

$$
\begin{aligned}
& \sum_{i=1}^{n} E\left[Z_{n2i}(\boldsymbol{\delta}) \right| X_i] \\
= \; & \sum_{i=1}^{n} \int_0^{v_{ni}(x)} \left(F_i \left(\frac{q(\alpha, X_i)\, \hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) + s \right) \right. \\
& \left. - F_i \left(\frac{q(\alpha, X_i)\, \hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \right) ds \\
= \; & \frac{1}{\sqrt{nh_q}} \sum_{i=1}^{n} \int_0^{\left[\delta_0 + h_q^{-1}(X_i - x)\delta_1\right] k\left(h_q^{-1}(X_i - x)\right) \hat{\tau}_n(X_i)} \\
& \left(F_i \left(\frac{q(\alpha, X_i)\, \hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) + \frac{u}{\sqrt{nh_q}} \right) \right. \\
& \left. - F_i \left(\frac{q(\alpha, X_i)\, \hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \right) du \\
= \; & \frac{1}{nh_q} \sum_{i=1}^{n} \int_0^{\left[\delta_0 + h_q^{-1}(X_i - x)\delta_1\right] k\left(h_q^{-1}(X_i - x)\right) \hat{\tau}_n(X_i)} \\
& u f_i \left(\frac{q(\alpha, X_i)\, \hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) du + o_p(1) \\
= \; & \frac{1}{2nh_q} \sum_{i=1}^{n} f_i \left(\frac{q(\alpha, X_i)\, \hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2} \hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \\
& \cdot \begin{bmatrix} \delta_0 & \delta_1 \end{bmatrix} \begin{bmatrix} 1 & h_q^{-1}(X_i - x) \\ h_q^{-1}(X_i - x) & h_q^{-2}(X_i - x)^2 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \end{bmatrix} \\
& \cdot k^2 \left(h_q^{-1}(X_i - x) \right) \tau_n^2 \left(\hat{g}_n(X_i) \right) + o_p(1).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \sum_{i=1}^{n} E\left[\bar{Z}_{n2i}(\boldsymbol{\delta}) \right| X_i] \\
= \; & \frac{1}{2nh_q} \sum_{i=1}^{n} f_i \left(q(\alpha, X_i) - \frac{1}{2} g(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \\
& \cdot \begin{bmatrix} \delta_0 & \delta_1 \end{bmatrix} \begin{bmatrix} 1 & h_q^{-1}(X_i - x) \\ h_q^{-1}(X_i - x) & h_q^{-2}(X_i - x)^2 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \end{bmatrix} \\
& \cdot k^2 \left(h_q^{-1}(X_i - x) \right) \tau_n^2 \left(g(X_i) \right) + o_p(1). \qquad (7)
\end{aligned}
$$

For each $\boldsymbol{\delta} \in \mathbb{R}^2$, $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some $\gamma \in (0,1)$ we have

$$
\begin{aligned}
& E\left[f_1\left(q\left(\alpha, X_1\right)-\frac{1}{2} g\left(X_1\right)\left(X_1-x\right)^2 q_g^{(2)}\left(\alpha, \xi_1\right)\right)\right. \\
& \cdot\left(\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}1 & h_q^{-1}\left(X_1-x\right) \\ h_q^{-1}\left(X_1-x\right) & h_q^{-2}\left(X_1-x\right)\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right]\right) \\
& \left.\cdot k^2\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^2\left(g\left(X_1\right)\right)\right] \\
= & \int f_1\left(q\left(\alpha, x_1\right)-\frac{1}{2} g\left(X_1\right)\left(X_1-x\right)^2 q_g^{(2)}\left(\alpha, \xi_1\right)\right) \\
& \cdot\left(\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}1 & h_q^{-1}\left(x_1-x\right) \\ h_q^{-1}\left(x_1-x\right) & h_q^{-2}\left(x_1-x\right)^2\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right]\right) \\
& \cdot k^2\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n^2\left(g\left(x_1\right)\right) g\left(x_1\right) d x_1 \\
= & h_q \int f_1\left(q\left(\alpha, x+h_q t_1\right)-\frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma h_q t_1+x\right)\right) \\
& \cdot\left(\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}1 & t_1 \\ t_1 & t_1^2\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right]\right) \\
& \cdot k^2\left(t_1\right) \tau_n^2\left(g\left(x+h_q t_1\right)\right) g\left(x+h_q t_1\right) d t_1 \\
= & h_q\left(\int f_1\left(q\left(\alpha, x+h_q t_1\right)\right)\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}1 & t_1 \\ t_1 & t_1^2\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right] k^2\left(t_1\right) \tau_n^2\left(g\left(x+h_q t_1\right)\right) g\left(x+h_q t_1\right) d t_1\right. \\
& \left.+O\left(h_q^2\right)\right) \\
= & h_q\left(\int f_{Y_1 \mid X_1=x}(q(\alpha, x))\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}1 & t_1 \\ t_1 & t_1^2\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right] k^2\left(t_1\right) \tau_n^2(g(x)) g(x) d t_1\right. \\
& \left.+O\left(h_q\right)\right) \\
= & h_q f_{Y_1 \mid X_1=x}(q(\alpha, x)) \tau_n^2(g(x)) g(x)\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right] \int\left[\begin{array}{cc}1 & t_1 \\ t_1 & t_1^2\end{array}\right] k^2\left(t_1\right) d t_1\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right]+O\left(h_q^2\right) \\
= & h_q f_{Y_1 \mid X_1=x}(q(\alpha, x)) g(x)\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right] \int\left[\begin{array}{cc}1 & t_1 \\ t_1 & t_1^2\end{array}\right] k^2\left(t_1\right) d t_1\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right]+O\left(n^{-\zeta} h_q\right) \\
& +O\left(h_q^2\right) \\
= & h_q f_{Y_1 \mid X_1=x}(q(\alpha, x)) g(x)\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right] \int\left[\begin{array}{cc}1 & t_1 \\ t_1 & t_1^2\end{array}\right] k^2\left(t_1\right) d t_1\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right]+O\left(h_q\left(n^{-\zeta}+h_q\right)\right).
\end{aligned}
$$

It follows from (7) that for each $\boldsymbol{\delta} \in \mathbb{R}^2$ and each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ that

$$
\begin{aligned}
& \sum_{i=1}^n E\left[\bar{Z}_{n 2 i}(\boldsymbol{\delta}) \mid X_i\right] \\
= & \frac{1}{2} f_{Y_1 \mid X_1=x}(q(\alpha, x)) g(x)\left[\begin{array}{ll}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}\int k^2\left(t_1\right) d t_1 & \int t_1 k^2\left(t_1\right) d t_1 \\ \int t_1 k^2\left(t_1\right) d t_1 & \int t_1^2 k^2\left(t_1\right) d t_1\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right] \\
& +o_p(1). \qquad (8)
\end{aligned}
$$

For each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ we have

$$\begin{aligned}
& \left| \sum_{i=1}^{n} E\left[Z_{n2i}(\boldsymbol{\delta}) \middle| X_i \right] - \sum_{i=1}^{n} E\left[\bar{Z}_{n2i}(\boldsymbol{\delta}) \middle| X_i \right] \right| \\
\leq \quad & \frac{1}{2nh_q} \sum_{i=1}^{n} \sup_{x \in \mathcal{X}} \left[\begin{array}{cc} \delta_0 & \delta_1 \end{array} \right] \left[\begin{array}{cc} 1 & h_q^{-1}(X_i - x) \\ h_q^{-1}(X_i - x) & h_q^{-2}(X_i - x)^2 \end{array} \right] \left[\begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] \\
& \cdot k^2\left(h_q^{-1}(X_i - x) \right) \\
& \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_i \left(\frac{q(\alpha, X_i)\hat{g}_n(X_i)}{g(X_i)} - \frac{1}{2}\hat{g}_n(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \tau_n^2(\hat{g}_n(X_i)) \right. \\
& \left. - f_i \left(q(\alpha, X_i) - \frac{1}{2} g(X_i)(X_i - x)^2 q_g^{(2)}(\alpha, \xi_i) \right) \tau_n^2(g(X_i)) \right| \\
& + o_p(1) \\
= \quad & \frac{1}{2nh_q} \sum_{i=1}^{n} \left[\begin{array}{cc} \delta_0 & \delta_1 \end{array} \right] \left[\begin{array}{cc} 1 & T_i \\ T_i & T_i^2 \end{array} \right] \left[\begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] k^2(T_i) \\
& \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_{Y_i | X_i = h_q T_i + x} \left(\frac{q(\alpha, h_q T_i + x)\hat{g}_n(h_q T_i + x)}{g(h_q T_i + x)} \right. \right. \\
& \left. - \frac{h_q^2}{2} \hat{g}_n(h_q T_i + x) T_i^2 q_g^{(2)}(\alpha, \gamma h_q T_i + x) \right) \\
& \cdot \tau_n^2(\hat{g}_n(h_q T_i + x)) \\
& - f_{Y_i | X_i = h_q T_i + x} \left(q(\alpha, h_q T_i + x) - \frac{h_q^2}{2} g(h_q T_i + x) T_i^2 q_g^{(2)}(\alpha, \gamma h_q T_i + x) \right) \\
& \left. \cdot \tau_n^2(g_n(h_q T_i + x)) \right| \\
& + o_p(1) \\
= \quad & \frac{1}{2nh_q} \sum_{i=1}^{n} \left[\begin{array}{cc} \delta_0 & \delta_1 \end{array} \right] \left[\begin{array}{cc} 1 & T_i \\ T_i & T_i^2 \end{array} \right] \left[\begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] k^2(T_i) \\
& \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_{Y_i | X_i = x + h_q T_i} \left(\frac{q(\alpha, x + h_q T_i)\hat{g}_n(x + h_q T_i)}{g(x + h_q T_i)} \right) \tau_n^2(\hat{g}_n(x + h_q T_i)) \right. \\
& + O\left(h_q^2 \right) \\
& \left. - f_{Y_i | X_i = x + h_q T_i} \left(q(\alpha, x + h_q T_i) \right) \tau_n^2(g(x + h_q T_i)) + O\left(h_q^2 \right) \right| \\
& + o_p(1) \\
= \quad & \frac{1}{2nh_q} \sum_{i=1}^{n} \left[\begin{array}{cc} \delta_0 & \delta_1 \end{array} \right] \left[\begin{array}{cc} 1 & T_i \\ T_i & T_i^2 \end{array} \right] \left[\begin{array}{c} \delta_0 \\ \delta_1 \end{array} \right] k^2(T_i) \\
& \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| f_{Y_1 | X_1 = x} \left(\frac{q(\alpha, x)\hat{g}_n(x)}{g(x)} \right) \tau_n^2(\hat{g}_n(x)) + O_p(h_q) \right. \\
& \left. - f_{Y_1 | X_1 = x} \left(q(\alpha, x) \right) \tau_n^2(g(x)) + O_p(h_q) \right| \\
& + o_p(1). \qquad (9)
\end{aligned}$$

From (4) and (5) we have

$$\begin{aligned}
& \sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|f_{Y_1|X_1=x}\left(\frac{q(\alpha,x)\hat{g}_n(x)}{g(x)}\right)\tau_n^2\left(\hat{g}_n(x)\right)-f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\tau_n^2\left(g(x)\right)\right| \\
= & \sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\tau_n^2\left(\hat{g}_n(x)\right)-f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\tau_n^2\left(g(x)\right)\right| \\
& +O_p\left(h_g^2+\frac{1}{\sqrt{nh_g}}\right) \\
= & O_p\left(n^{\zeta}\left(h_g^2+\frac{1}{\sqrt{nh_g}}\right)\right).
\end{aligned}$$

It follows from (9) that

$$\begin{aligned}
& \left|\sum_{i=1}^{n}E\left[\left.Z_{n2i}(\boldsymbol{\delta})\right|X_i\right]-\sum_{i=1}^{n}E\left[\left.\bar{Z}_{n2i}(\boldsymbol{\delta})\right|X_i\right]\right| \\
= & O_p\left(\frac{1}{\sqrt{nh_q}}+n^{\zeta}\left(h_g^2+\frac{1}{\sqrt{nh_g}}\right)+h_q\right) \\
= & o_p(1) \qquad (10)
\end{aligned}$$

for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$. Here the assumptions $\sup_{x\in\mathcal{X}}g(x)<\infty$, $\int t_1^4k^4\left(t_1\right)dt_1<\infty$ and

$$n^{\zeta}\left(h_g^2+\frac{1}{\sqrt{nh_g}}\right)=o(1),$$

amongst others, have been invoked.

Combine (10) with (8) to deduce that for each $\boldsymbol{\delta}\in\mathbb{R}^2$ and each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$,

$$\begin{aligned}
& \sum_{i=1}^{n}E\left[\left.Z_{n2i}(\boldsymbol{\delta})\right|X_i\right] \\
= & \frac{1}{2}f_{Y_1|X_1=x}\left(q\left(\alpha,x\right)\right)g\left(x\right)\left[\begin{array}{cc}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc}\int k^2\left(t_1\right)dt_1 & \int t_1k^2\left(t_1\right)dt_1 \\ \int t_1k^2\left(t_1\right)dt_1 & \int t_1^2k^2\left(t_1\right)dt_1\end{array}\right]\left[\begin{array}{c}\delta_0 \\ \delta_1\end{array}\right] \\
& +o_p(1).
\end{aligned}$$

Next, note that

$$\begin{aligned}
& \sum_{i=1}^{n}Var\left[\left.Z_{n2i}(\boldsymbol{\delta})\right|X_i\right] \\
\leq & \frac{1}{\sqrt{nh_q}}\max_i\left(\delta_0+h_q^{-1}\left(X_i-x\right)\delta_1\right)k\left(h_q^{-1}\left(X_i-x\right)\right)\tau_n\left(\hat{g}_n\left(X_i\right)\right) \\
& \cdot\sum_{i=1}^{n}E\left[\left.Z_{n2i}(\boldsymbol{\delta})\right|X_i\right] \\
= & \frac{1}{\sqrt{nh_q}}\max_i\left(\delta_0+h_q^{-1}\left(X_i-x\right)\delta_1\right)k\left(h_q^{-1}\left(X_i-x\right)\right)\tau_n\left(g\left(X_i\right)\right) \\
& \cdot\sum_{i=1}^{n}E\left[\left.\bar{Z}_{n2i}\left(\boldsymbol{\delta}\right)\right|X_i\right] \\
& +o_p(1) \qquad (11)
\end{aligned}$$

for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$ by virtue of (5) and (10).

We have

$$\begin{aligned}
& E\left[\frac{1}{\sqrt{nh_q}}\left(\delta_0+h_q^{-1}\left(X_1-x\right)\delta_1\right)k\left(h_q^{-1}\left(X_1-x\right)\right)\tau_n\left(g\left(X_i\right)\right)\right] \\
= & \sqrt{\frac{h_q}{n}}\int\left(\delta_0+\delta_1 t_1\right)k\left(t_1\right)\tau_n\left(g\left(x+h_q t_1\right)\right)g\left(x+h_q t_1\right)dt_1 \\
\leq & \sqrt{\frac{h_q}{n}}g(x)\delta_0+O\left(\sqrt{\frac{h_q}{n}}\cdot h_q^2\right),
\end{aligned}$$

and

$$Var\left[\frac{1}{\sqrt{nh_q}}\left(\delta_0+h_q^{-1}\left(X_1-x\right)\delta_1\right)k\left(h_q^{-1}\left(X_1-x\right)\right)\tau_n\left(g\left(X_1\right)\right)\right]=O\left(\frac{1}{n}\right),$$

so

$$\begin{aligned}
\frac{1}{\sqrt{nh_q}}\max_i\left[\delta_0+h_q^{-1}\left(X_i-x\right)\delta_1\right]k\left(h_q^{-1}\left(X_i-x\right)\right)\tau_n\left(g\left(X_i\right)\right) &= O_p\left(\sqrt{\frac{h_q}{n}}+\frac{1}{\sqrt{n}}\right) \\
&= o_p(1).
\end{aligned}$$

Combine this with (11) to deduce that

$$\sum_{i=1}^{n}\left(Z_{n2i}(\boldsymbol{\delta})-E\left[\left.Z_{n2i}(\boldsymbol{\delta})\right|X_i\right]\right)=o_p(1)$$

for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$ and each $\boldsymbol{\delta}\in\mathbb{R}^2$.

It follows that for each $\boldsymbol{\delta}\in\mathbb{R}^2$ and $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$ we have

$$\begin{aligned}
Z_{n2}(\boldsymbol{\delta}) = & \frac{1}{2}f_{Y_1|X_1=x}\left(q\left(\alpha,x\right)\right)g\left(x\right)\begin{bmatrix}\delta_0 & \delta_1\end{bmatrix}\begin{bmatrix}\int k^2\left(t_1\right)dt_1 & \int t_1k^2\left(t_1\right)dt_1 \\ \int t_1k^2\left(t_1\right)dt_1 & \int t_1^2k^2\left(t_1\right)dt_1\end{bmatrix}\begin{bmatrix}\delta_0 \\ \delta_1\end{bmatrix} \\
& +o_p(1). \qquad (12)
\end{aligned}$$

Next, consider the asymptotic behaviour of $Z_{n1}(\boldsymbol{\delta})$ on $\mathcal{A}\times\mathcal{X}$. Recall the definition of $\bar{u}_i^*(x)$ in (6) and define

$$\bar{Z}_{n1}(\boldsymbol{\delta})\equiv-\frac{1}{\sqrt{nh_q}}\sum_{i=1}^{n}\left[\delta_0+h_q^{-1}\left(X_i-x\right)\delta_1\right]\psi_\alpha\left(\bar{u}_i^*(x)\right)k\left(h_q^{-1}\left(X_i-x\right)\right)\tau_n\left(g\left(X_i\right)\right).$$

Then

$$\begin{aligned}
& \left|Z_{n1}(\boldsymbol{\delta})-\bar{Z}_{n1}(\boldsymbol{\delta})\right| \\
\leq & \frac{1}{\sqrt{nh_q}}\sum_{i=1}^{n}\sup_{x\in\mathcal{X}}\left|\left[\delta_0+h_q^{-1}\left(X_i-x\right)\delta_1\right]k\left(h_q^{-1}\left(X_i-x\right)\right)\right| \\
& \cdot\left|\tau_n\left(g\left(X_i\right)\right)-\tau_n\left(\hat{g}_n\left(X_i\right)\right)\right| \\
= & \frac{1}{\sqrt{nh_q}}\sum_{i=1}^{n}\left|\delta_0+T_i\delta_1\right|k\left(T_i\right)\sup_{x\in\mathcal{X}}\left|\tau_n\left(g\left(x+h_qT_i\right)\right)-\tau_n\left(\hat{g}_n\left(x+h_qT_i\right)\right)\right| \\
= & \frac{1}{\sqrt{nh_q}}\sum_{i=1}^{n}\left|\delta_0+T_i\delta_1\right|k\left(T_i\right)\sup_{x\in\mathcal{X}}\left|\tau_n\left(\hat{g}_n\left(x\right)\right)-\tau_n\left(g(x)\right)+O_p\left(h_q\right)\right|.
\end{aligned}$$

From (5) we have

$$\sup_{x\in\mathcal{X}}\left|\tau_n\left(\hat{g}_n(x)\right)-\tau_n\left(g(x)\right)\right|=O_p\left(n^\zeta\left(h_g^2+\frac{1}{\sqrt{nh_g}}\right)\right).$$

Assuming that $n^\zeta\left(h_g^2+1/\sqrt{nh_g}\right)=o(1)$ and invoking the various conditions of Assumptions A.3 and A.4 on $g(\cdot)$ and $k(\cdot)$, we have

$$\left|Z_{n1}(\boldsymbol{\delta})-\bar{Z}_{n1}(\boldsymbol{\delta})\right|=o_p(1) \qquad (13)$$

for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$.

The remainder of the proof proceeds according to two steps, which will be discussed in sequence. The first step implies the convergence in distribution of $\bar{Z}_{n1}(\boldsymbol{\delta})$ for each $\boldsymbol{\delta} \in \mathbb{R}^2$ and each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$. The second step ensures that for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$, the estimator $\hat{q}_n^*(\alpha, x)$ has a bias that vanishes as $n \to \infty$. Taken together, the results of these two steps imply an asymptotic representation for $\hat{q}_n^*(\alpha, x)$ that holds pointwise on $\mathcal{A} \times \mathcal{X}$, and from which the $\sqrt{nh_q}$-consistency of $\hat{q}_n^*(\alpha, x)$ for $q(\alpha, x)/g(x)$ for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ can be deduced.

The two steps are described as follows. In particular, they involve showing:

(1) That

$$\begin{aligned} & \frac{1}{\sqrt{nh_q}} \sum_{i=1}^{n} \left(\psi_\alpha\left(\bar{u}_i^*(x)\right) - E\left[\psi_\alpha\left(\bar{u}_i^*(x)\right) \middle| X_i\right]\right) \begin{bmatrix} 1 \\ h_q^{-1}\left(X_i - x\right) \end{bmatrix} k\left(h_q^{-1}\left(X_i - x\right)\right) \\ & \cdot \tau_n\left(g\left(X_i\right)\right) \\ \xrightarrow{d} \; & N\left(\mathbf{0}, \boldsymbol{\Sigma}(\alpha, x)\right) \end{aligned}$$

for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$, where $\boldsymbol{\Sigma}(\alpha, x)$ is a symmetric and positive-definite matrix for which a closed-form representation will be derived.

(2) That

$$\sqrt{\frac{n}{h_q}} E\left[\psi_\alpha\left(\bar{u}_i^*(x)\right) \begin{bmatrix} 1 \\ h_q^{-1}\left(X_1 - x\right) \end{bmatrix} k\left(h_q^{-1}\left(X_1 - x\right)\right) \tau_n\left(g\left(X_i\right)\right)\right] = o(1)$$

pointwise on $\mathcal{A} \times \mathcal{X}$. Moreover, the rate of convergence is sufficiently fast for the bias of the estimator $\hat{q}_n^*(\alpha, x)$ to vanish in large samples.

2.2 First step: Convergence in finite-dimensional distributions

If the conditions of Liapounov's central limit theorem are met, we have for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ that

$$\begin{aligned} & \frac{1}{\sqrt{nh_q}} \sum_{i=1}^{n} \left(\psi_\alpha\left(\bar{u}_i^*(x)\right) - E\left[\psi_\alpha\left(\bar{u}_i^*(x)\right) \middle| X_i\right]\right) \begin{bmatrix} 1 \\ h_q^{-1}\left(X_i - x\right) \end{bmatrix} k\left(h_q^{-1}\left(X_i - x\right)\right) \\ & \cdot \tau_n\left(g\left(X_i\right)\right) \\ \xrightarrow{d} \; & N\left(\mathbf{0}, \boldsymbol{\Sigma}(\alpha, x)\right), \end{aligned} \tag{14}$$

where

$$\begin{aligned} & \boldsymbol{\Sigma}(\alpha, x) \\ = \; & \lim_{n \to \infty} E\left[h_q^{-1} Var\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \middle| X_1\right] \begin{bmatrix} 1 & h_q^{-1}\left(X_1 - x\right) \\ h_q^{-1}\left(X_1 - x\right) & h_q^{-2}\left(X_1 - x\right)^2 \end{bmatrix} \right. \\ & \left. \cdot k^2\left(h_q^{-1}\left(X_1 - x\right)\right) \tau_n^2\left(g\left(X_1\right)\right)\right]. \end{aligned}$$

A closed-form representation for $\boldsymbol{\Sigma}(\alpha, x)$ is first derived. Let $T_1 \equiv h_q^{-1}\left(X_1 - x\right)$, so that $X_1 = x + h_q T_1$. For some ξ_1 on a line segment between x and X_1 we have

$$\begin{aligned} & \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} \\ = \; & \frac{q\left(\alpha, x + h_q T_1\right)}{g\left(x + h_q T_1\right)} \\ = \; & \frac{q(\alpha, x)}{g(x)} + h_q T_1 q_g^{(1)}(\alpha, x) + \frac{h_q^2}{2} T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right). \end{aligned}$$

We have

$$
\begin{aligned}
& E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right)\middle| X_1\right] \\
= \;& \alpha - P\left[\frac{Y_1}{g\left(X_1\right)} \leq \frac{q(\alpha, x)}{g(x)} + \left(X_1 - x\right) q_g^{(1)}(\alpha, x) \middle| X_1\right] \\
= \;& \alpha - P\left[\frac{Y_1}{g\left(X_1\right)} \leq \frac{q(\alpha, x)}{g(x)} + h_q T_1 q_g^{(1)}(\alpha, x) \middle| X_1\right] \\
= \;& \alpha - P\left[\frac{Y_1}{g\left(X_1\right)} \leq \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} - \frac{h_q^2}{2} T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) \middle| X_1\right] \\
= \;& \alpha - \left(F_1\left(q\left(\alpha, X_1\right)\right) - \frac{h_q^2}{2} g\left(X_1\right) T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right)\right) \\
= \;& \frac{h_q^2}{2} g\left(X_1\right) T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right) \qquad (15)
\end{aligned}
$$

for some $\zeta_1\left(\alpha, h_q\right)$ in the interval $\left(q\left(\alpha, X_1\right) - h_q^2/2g\left(X_1\right) T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right), q\left(\alpha, X_1\right)\right)$.

As such,

$$
\begin{aligned}
& Var\left[\psi_\alpha\left(\bar{u}_1^*(x)\right)\middle| X_1\right] \\
= \;& E\left[\psi_\alpha^2\left(\bar{u}_1^*(x)\right)\middle| X_1\right] - \left(E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right)\middle| X_1\right]\right)^2 \\
= \;& (\alpha - 1)^2 P\left[\frac{Y_1}{g\left(X_1\right)} \leq \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} - \frac{h_q^2}{2} T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) \middle| X_1\right] \\
& + \alpha^2 P\left[\frac{Y_1}{g\left(X_1\right)} > \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} - \frac{h_q^2}{2} T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) \middle| X_1\right] \\
& - \frac{h_q^4}{4} g^2\left(X_1\right) T_1^4 \left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^2 f_1^2\left(\zeta_1\left(\alpha, h_q\right)\right) \\
= \;& (\alpha - 1)^2 \left(\alpha - \frac{h_q^2}{2} g\left(X_1\right) T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right)\right) \\
& + \alpha^2 \left(1 - \alpha + \frac{h_q^2}{2} g\left(X_1\right) T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right)\right) \\
& - \frac{h_q^4}{4} g^2\left(X_1\right) T_1^4 \left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^2 f_1^2\left(\zeta_1\left(\alpha, h_q\right)\right) \\
= \;& \alpha(1 - \alpha) + h_q^2 \cdot \frac{2\alpha - 1}{2} g\left(X_1\right) T_1^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right) \\
& - \frac{h_q^4}{4} g^2\left(X_1\right) T_1^4 \left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^2 f_1^2\left(\zeta_1\left(\alpha, h_q\right)\right). \qquad (16)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& E\left[h_q^{-1} Var\left[\psi_\alpha\left(\bar{u}_1^*(x)\right)\middle| X_1\right] k^2\left(h_q^{-1}\left(X_1 - x\right)\right) \tau_n^2\left(g\left(X_1\right)\right)\right] \\
= \;& \alpha(1 - \alpha) E\left[h_q^{-1} k^2\left(h_q^{-1}\left(X_1 - x\right)\right) \tau_n^2\left(g\left(X_1\right)\right)\right] \\
& + \frac{2\alpha - 1}{2} E\left[h_q^{-1} g\left(X_1\right) T_1^2 h_q^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right) k^2\left(h_q^{-1}\left(X_1 - x\right)\right) \tau_n^2\left(g\left(X_1\right)\right)\right] \\
& - \frac{1}{4} E\left[h_q^{-1} g^2\left(X_1\right) T_1^4 h_q^4 \left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^2 f_1^2\left(\zeta_1\left(\alpha, h_q\right)\right) k^2\left(h_q^{-1}\left(X_1 - x\right)\right) \tau_n^2\left(g\left(X_1\right)\right)\right] \\
\equiv \;& \Sigma_{111} + \Sigma_{112} + \Sigma_{113}.
\end{aligned}
$$

We have for ξ_1 an intermediate point between X_1 and x that

$$\begin{aligned}
\Sigma_{111} &= \alpha(1-\alpha)h_q^{-1}\int k^2\left(h_q^{-1}(x_1-x)\right)\tau_n^2\left(g\left(x_1\right)\right)g\left(x_1\right)dx_1 \\
&= \alpha(1-\alpha)\int k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_qt_1\right)\right)g\left(x+h_qt_1\right)dt_1 \\
&= \alpha(1-\alpha)\int k^2\left(t_1\right) \\
&\quad \cdot\left[\tau_n^2\left(g(x)\right)g(x)+h_qt_1h_q\left(2\tau_n\left(g\left(\xi_1\right)\right)\tau_n^{(1)}\left(g\left(\xi_1\right)\right)g^{(1)}\left(\xi_1\right)g\left(\xi_1\right)+\tau_n^2\left(g\left(\xi_1\right)\right)g^{(1)}\left(\xi_1\right)\right)\right]dt_1 \\
&= \alpha(1-\alpha)\tau_n^2\left(g(x)\right)g(x)\int k^2\left(t_1\right)dt_1+O\left(h_q\right) \\
&= \alpha(1-\alpha)g(x)\int k^2\left(t_1\right)dt_1+\alpha(1-\alpha)\left(\tau_n^2\left(g(x)\right)-1\right)g(x)\int k^2\left(t_1\right)dt_1+O\left(h_q\right) \\
&= \alpha(1-\alpha)g(x)\int k^2\left(t_1\right)dt_1+O\left(n^{-\zeta}+h_q\right) \\
&= \alpha(1-\alpha)g(x)\int k^2\left(t_1\right)dt_1+o(1).
\end{aligned}$$

Now consider that for fixed $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$ and some constants $(\gamma_1,\gamma_2)\in(0,1)^2$

$$\begin{aligned}
\Sigma_{112} &= \frac{2\alpha-1}{2}h_q^{-1}\int g^2\left(x_1\right)\left(x_1-x\right)^2q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right) \\
&\quad \cdot f_1\left(q\left(\alpha,x_1\right)-\gamma_2\cdot\frac{1}{2}g\left(x_1\right)\left(x_1-x\right)^2q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right)\right) \\
&\quad \cdot k^2\left(h_q^{-1}\left(x_1-x\right)\right)\tau_n^2\left(g\left(x_1\right)\right)dx_1 \\
&= h_q^2\cdot\frac{2\alpha-1}{2}\int g^2\left(x+h_qt_1\right)t_1^2q_g^{(2)}\left(\alpha,\gamma_1h_qt_1+x\right) \\
&\quad \cdot f_1\left(q\left(\alpha,x+h_qt_1\right)-\gamma_2\cdot\frac{h_q^2}{2}g\left(x+h_qt_1\right)t_1^2q_g^{(2)}\left(\alpha,\gamma_1h_qt_1+x\right)\right) \\
&\quad \cdot k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_qt_1\right)\right)dt_1 \\
&= \frac{2\alpha-1}{2}\left(h_q^2\int g^2\left(x+h_qt_1\right)t_1^2q_g^{(2)}\left(\alpha,\gamma_1h_qt_1+x\right)\right. \\
&\quad \cdot f_1\left(q\left(\alpha,x+h_qt_1\right)\right)k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_qt_1\right)\right)dt_1 \\
&\quad \left.+O\left(h_q^4\right)\right) \\
&= \frac{2\alpha-1}{2}\left(h_q^2\int g^2(x)t_1^2q_g^{(2)}(\alpha,x)f_{Y_1|X_1=x}\left(q(\alpha,x)\right)k^2\left(t_1\right)\tau_n^2\left(g(x)\right)dt_1\right. \\
&\quad \left.+O\left(h_q^3\right)\right) \\
&\leq \sup_{x\in\mathcal{X}}g^2(x)\cdot\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q_g^{(2)}(\alpha,x)\right|\cdot\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}f_{Y_1|X_1=x}\left(q(\alpha,x)\right) \\
&\quad \cdot\frac{2\alpha-1}{2}\left(h_q^2\int t_1^2k^2\left(t_1\right)dt_1+O\left(h_q^3\right)\right) \\
&= O\left(h_q^2\right) \\
&= o(1),
\end{aligned}$$

where amongst others, the conditions that

$$\begin{aligned}
\sup_{x\in\mathcal{X}}g(x) &< \infty, \\
\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}f_{Y_1|X_1=x}\left(q(\alpha,x)\right) &< \infty, \\
\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q_g^{(2)}(\alpha,x)\right| &< \infty
\end{aligned}$$

and

$$\int t_1^4 k^2(t_1)\,dt_1 < \infty$$

have been invoked.

Similarly, we have for fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some constants $(\gamma_1, \gamma_2) \in (0,1)^2$ that

$$\begin{aligned}
\Sigma_{113} &= -\frac{1}{4} h_q^{-1} \int g^3(x_1)(x_1 - x)^4 \left| q_g^{(2)}(\alpha, \gamma_1(x_1 - x) + x) \right|^2 \\
&\quad \cdot f_1^2 \left(q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1)(x_1 - x)^2 \left| q_g^{(2)}(\alpha, \gamma_1(x_1 - x) + x) \right|^2 \right) \\
&\quad \cdot k^2\left(h_q^{-1}(x_1 - x)\right) \tau_n^2(g(x_1))\,dx_1 \\
&= -\frac{h_q^4}{4} \int g^3(x + h_q t_1) t_1^4 \left(q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right)^2 \\
&\quad \cdot f_1^2 \left(q(\alpha, x + h_q t_1) - \gamma_2 \cdot \frac{h_q^2}{2} g(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x) \right) \\
&\quad \cdot k^2(t_1) \tau_n^2(g(x + h_q t_1))\,dt_1 \\
&= -\frac{1}{4} \left(h_q^4 g^3(x) \left(q_g^{(2)}(\alpha, x) \right)^2 f_{Y_1|X_1=x}^2(q(\alpha, x)) \tau_n^2(g(x)) \int t_1^4 k^2(t_1)\,dt_1 \right. \\
&\quad \left. + O\left(h_q^5\right) \right) \\
&\le \sup_{x \in \mathcal{X}} g^3(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left(q_g^{(2)}(\alpha, x) \right)^2 \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}^2(q(\alpha, x)) \\
&\quad \cdot \frac{1}{4} \left(h_q^4 \int t_1^4 k^2(t_1)\,dt_1 + O\left(h_q^5\right) \right) \\
&= O\left(h_q^4\right) \\
&= o(1),
\end{aligned}$$

where amongst others, the conditions that

$$\begin{aligned}
\sup_{x \in \mathcal{X}} g(x) &< \infty, \\
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}(q(\alpha, x)) &< \infty, \\
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| q_g^{(2)}(\alpha, x) \right| &< \infty
\end{aligned}$$

and

$$\int t_1^8 k^2(t_1)\,dt_1 < \infty$$

have been invoked.

Next, consider

$$\begin{aligned}
&E\left[h_q^{-1} Var\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \middle| X_1 \right] k^2\left(h_q^{-1}(x_1 - x)\right) \tau_n^2(g(X_1)) h_q^{-1}(x_1 - x) \right] \\
= \; &\alpha(1-\alpha) E\left[h_q^{-1} k^2\left(h_q^{-1}(x_1 - x)\right) \tau_n^2(g(X_1)) h_q^{-1}(X_1 - x) \right] \\
&+ \frac{2\alpha - 1}{2} E\left[h_q^{-1} g(X_1) \cdot T_1^2 h_q^2 q_g^{(2)}(\alpha, \xi_1) \cdot f_1(\zeta_1(\alpha, h_q)) \right. \\
&\left. \cdot k^2\left(h_q^{-1}(X_1 - x)\right) \tau_n^2(g(X_1)) h_q^{-1}(X_1 - x) \right] \\
&- \frac{1}{4} E\left[h_q^{-1} g^2(X_1) \left(h_q^2 T_1^2 q_g^{(2)}(\alpha, \xi_1) \right)^2 f_1^2(\zeta_1(\alpha, h_q)) \right. \\
&\left. \cdot k^2\left(h_q^{-1}(X_1 - x)\right) \tau_n^2(g(X_1)) h_q^{-1}(X_1 - x) \right] \\
\equiv \; &\Sigma_{121} + \Sigma_{122} + \Sigma_{123}.
\end{aligned}$$

We have

$$
\begin{aligned}
\Sigma_{121} &= \alpha(1-\alpha)h_q^{-1}\int k^2\left(h_q^{-1}(x_1-x)\right)\tau_n^2\left(g\left(x_1\right)\right)h_q^{-1}\left(x_1-x\right)g\left(x_1\right)dx_1\\
&= \alpha(1-\alpha)\int k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_q t_1\right)\right)t_1 g\left(x+h_q t_1\right)dt_1\\
&= \alpha(1-\alpha)g(x)\int t_1 k^2\left(t_1\right)dt_1+O\left(n^{-\zeta}+h_q\right)\\
&= \alpha(1-\alpha)g(x)\int t_1 k^2\left(t_1\right)dt_1+o(1).
\end{aligned}
$$

For fixed $(\alpha, x) \in \mathcal{A}\times\mathcal{X}$ and some constants $(\gamma_1,\gamma_2)\in(0,1)^2$,

$$
\begin{aligned}
\Sigma_{122} &= \frac{2\alpha-1}{2}h_q^{-1}\int g^2\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right)\\
&\quad\cdot f_1\left(q\left(\alpha,x_1\right)-\gamma_2\cdot\frac{1}{2}g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right)\right)\\
&\quad\cdot k^2\left(h_q^{-1}\left(x_1-x\right)\right)\tau_n^2\left(g\left(x_1\right)\right)\cdot h_q^{-1}\left(x_1-x\right)dx_1\\
&= \frac{2\alpha-1}{2}\cdot h_q^2\int g^2\left(x+h_q t_1\right)t_1^2 q_g^{(2)}\left(\alpha,\gamma_1 h_q t_1+x\right)\\
&\quad\cdot f_1\left(q\left(\alpha,h_q t_1+x\right)-\gamma_2\cdot\frac{1}{2}g\left(x+h_q t_1\right)t_1^2 h_q^2 q_g^{(2)}\left(\alpha,\gamma_1 h_q t_1+x\right)\right)\\
&\quad\cdot k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_q t_1\right)\right)\cdot t_1 dt_1\\
&= \frac{2\alpha-1}{2}\left(h_q^2\int g^2(x)t_1^2 q_g^{(2)}(\alpha,x)f_{Y_1|X_1=x}\left(q(\alpha,x)\right)k^2\left(t_1\right)\tau_n^2\left(g(x)\right)t_1 dt_1\right.\\
&\quad\left.+O\left(h_q^3\right)\right)\\
&\leq \sup_{x\in\mathcal{X}}g^2(x)\cdot\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q_g^{(2)}(\alpha,x)\right|\cdot\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\\
&\quad\cdot\frac{2\alpha-1}{2}\left(h_q^2\int t_1^2 k^2\left(t_1\right)t_1 dt_1+O\left(h_q^3\right)\right)\\
&= O\left(h_q^2\right)\\
&= o(1),
\end{aligned}
$$

where amongst others, the conditions that

$$
\begin{aligned}
\sup_{x\in\mathcal{X}}g(x) &< \infty,\\
\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}f_{Y_1|X_1=x}\left(q(\alpha,x)\right) &< \infty,\\
\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q_g^{(2)}(\alpha,x)\right| &< \infty
\end{aligned}
$$

and

$$
\int t_1^6 k^2\left(t_1\right)dt_1<\infty
$$

have been invoked.

Similarly, we have for fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some constants $(\gamma_1, \gamma_2) \in (0,1)^2$ that

$$\begin{aligned}
\Sigma_{123} &= -\frac{1}{4} h_q^{-1} \int g^3(x_1)(x_1 - x)^4 \left(q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x)\right)^2 \\
&\quad \cdot f_1^2 \left(q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1)(x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1 (x_1 - x) + x)\right) \\
&\quad \cdot k^2 \left(h_q^{-1}(x_1 - x)\right) \tau_n^2 (g(x_1)) \cdot h_q^{-1}(x_1 - x) \, dx_1 \\
&= -\frac{1}{4} \cdot h_q^4 \int g^3(x + h_q t_1) t_1^4 \left(q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x)\right)^2 \\
&\quad \cdot f_1^2 \left(q(\alpha, h_q t_1 + x) - \gamma_2 \cdot \frac{h_q^2}{2} g(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x)\right) \\
&\quad \cdot k^2(t_1) \tau_n^2 (g(x + h_q t_1)) \cdot t_1 dt_1 \\
&= -\frac{h_q^4}{4} \left(\int g^3(x) t_1^4 \left(q_g^{(2)}(\alpha, x)\right)^2 f_{Y_1|X_1=x}^2 (q(\alpha, x)) k^2(t_1) \tau_n^2(g(x)) t_1 dt_1 \right. \\
&\quad \left. + O\left(h_q^5\right)\right) \\
&\leq \sup_{x \in \mathcal{X}} g^3(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| q_g^{(2)}(\alpha, x) \right| \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}^2 (q(\alpha, x)) \\
&\quad \cdot \frac{1}{4} \left(h_q^4 \int t_1^4 k^2(t_1) t_1 dt_1 + O\left(h_q^5\right) \right) \\
&= O\left(h_q^4\right) \\
&= o(1),
\end{aligned}$$

where amongst others, the conditions that

$$\begin{aligned}
\sup_{x \in \mathcal{X}} g(x) &< \infty, \\
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}(q(\alpha, x)) &< \infty, \\
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| q_g^{(2)}(\alpha, x) \right| &< \infty
\end{aligned}$$

and

$$\int t_1^{10} k^2(t_1) \, dt_1 < \infty$$

have been invoked.

Lastly, take

$$\begin{aligned}
& E\left[h_q^{-1} Var\left[\psi_\alpha(\bar{u}_1^*(x)) | X_1\right] k^2\left(h_q^{-1}(X_1 - x)\right) \tau_n^2(g(X_1)) h_q^{-2}(X_1 - x)^2\right] \\
= \; & \alpha(1-\alpha) E\left[h_q^{-1} k^2\left(h_q^{-1}(X_1 - x)\right) \tau_n^2(g(X_1)) h_q^{-2}(X_1 - x)^2\right] \\
& + \frac{2\alpha - 1}{2} E\left[h_q^{-1} g(X_1) T_1^2 h_q^2 q_g^{(2)}(\alpha, \xi_1) f_1(\zeta_1(\alpha, h_q)) k^2\left(h_q^{-1}(X_1 - x)\right) \tau_n^2(g(X_1)) \right. \\
& \left. \cdot h_q^{-2}(X_1 - x)^2\right] \\
& - \frac{1}{4} E\left[h_q^{-1} g^2(X_1) T_1^4 h_q^4 \left(q_g^{(2)}(\alpha, \xi_1)\right)^2 f_1^2(\zeta_1(\alpha, h_q)) k^2\left(h_q^{-1}(X_1 - x)\right) \tau_n^2(g(X_1)) \right. \\
& \left. \cdot h_q^{-2}(X_1 - x)^2\right] \\
\equiv \; & \Sigma_{221} + \Sigma_{222} + \Sigma_{223}.
\end{aligned}$$

We have

$$\begin{aligned}
\Sigma_{221} &= \alpha(1-\alpha)h_q^{-1}\int k^2\left(h_q^{-1}\left(x_1-x\right)\right)\tau_n^2\left(g\left(x_1\right)\right)h_q^{-2}\left(x_1-x\right)^2 g\left(x_1\right)dx_1 \\
&= \alpha(1-\alpha)\int k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_qt_1\right)\right)t_1^2 g\left(x+h_qt_1\right)dt_1 \\
&= \alpha(1-\alpha)g(x)\int t_1^2k^2\left(t_1\right)dt_1+O\left(n^{-\zeta}+h_q\right) \\
&= \alpha(1-\alpha)g(x)\int t_1^2k^2\left(t_1\right)dt_1+o(1).
\end{aligned}$$

For fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some constants $(\gamma_1, \gamma_2) \in (0,1)^2$,

$$\begin{aligned}
\Sigma_{222} &= \frac{2\alpha-1}{2}\cdot h_q^{-1}\int g^2\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right) \\
&\quad \cdot f_1\left(q\left(\alpha,x_1\right)-\gamma_2\cdot\frac{1}{2}g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right)\right) \\
&\quad \cdot k^2\left(h_q^{-1}\left(x_1-x\right)\right)\tau_n^2\left(g\left(x_1\right)\right)\cdot h_q^{-2}\left(x_1-x\right)^2dx_1 \\
&= \frac{2\alpha-1}{2}\cdot h_q^2\int g^2\left(x+h_qt_1\right)t_1^2q_g^{(2)}\left(\alpha,\gamma_1h_qt_1+x\right) \\
&\quad \cdot f_1\left(q\left(\alpha,h_qt_1+x\right)-\gamma_2\cdot\frac{1}{2}g\left(x+h_qt_1\right)t_1^2h_q^2q_g^{(2)}\left(\alpha,\gamma_1h_qt_1+x\right)\right) \\
&\quad \cdot k^2\left(t_1\right)\tau_n^2\left(g\left(x+h_qt_1\right)\right)\cdot t_1^2dt_1 \\
&= \frac{2\alpha-1}{2}\left(h_q^2\int g^2(x)t_1^4q_g^{(2)}(\alpha,x)f_{Y_1|X_1=x}(q(\alpha,x))k^2\left(t_1\right)\tau_n^2\left(g(x)\right)dt_1\right. \\
&\quad \left.+O\left(h_q^3\right)\right) \\
&\leq \sup_{x\in\mathcal{X}}g^2(x)\cdot\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q_g^{(2)}(\alpha,x)\right|\cdot\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}f_{Y_1|X_1=x}(q(\alpha,x)) \\
&\quad \cdot\frac{2\alpha-1}{2}\left(h_q^2\int t_1^4k^2\left(t_1\right)dt_1+O\left(h_q^3\right)\right) \\
&= O\left(h_q^2\right) \\
&= o(1),
\end{aligned}$$

where amongst others, the conditions that

$$\begin{aligned}
\sup_{x\in\mathcal{X}}g(x) &< \infty, \\
\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}f_{Y_1|X_1=x}(q(\alpha,x)) &< \infty, \\
\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q_g^{(2)}(\alpha,x)\right| &< \infty
\end{aligned}$$

and

$$\int t_1^8k^2\left(t_1\right)dt_1 < \infty$$

have been invoked.

Similarly, we have for fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some constants $(\gamma_1, \gamma_2) \in (0,1)^2$ that

$$\begin{aligned}
\Sigma_{223} &= -\frac{1}{4} h_q^{-1} \int g^3(x_1)(x_1 - x)^4 \left(q_g^{(2)}(\alpha, \gamma_1(x_1 - x) + x)\right)^2 \\
&\quad \cdot f_1^2 \left(q(\alpha, x_1) - \gamma_2 \cdot \frac{1}{2} g(x_1)(x_1 - x)^2 q_g^{(2)}(\alpha, \gamma_1(x_1 - x) + x)\right) \\
&\quad \cdot k^2\left(h_q^{-1}(x_1 - x)\right) \tau_n^2(g(x_1)) \cdot h_q^{-2}(x_1 - x)^2 dx_1 \\
&= -\frac{1}{4} \cdot h_q^4 \int g^3(x + h_q t_1) t_1^4 \left(q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x)\right)^2 \\
&\quad \cdot f_1^2 \left(q(\alpha, h_q t_1 + x) - \gamma_2 \cdot \frac{h_q^2}{2} g(x + h_q t_1) t_1^2 q_g^{(2)}(\alpha, \gamma_1 h_q t_1 + x)\right) \\
&\quad \cdot k^2(t_1) \tau_n^2(g(x + h_q t_1)) \cdot t_1^2 dt_1 \\
&= -\frac{1}{4} \left(h_q^4 \int g^3(x) t_1^4 \left(q_g^{(2)}(\alpha, x)\right)^2 f_{Y_1|X_1=x}^2 (q(\alpha, x)) k^2(t_1) \tau_n^2(g(x)) t_1^2 dt_1 \right. \\
&\quad \left. + O\left(h_q^5\right)\right) \\
&\leq \sup_{x \in \mathcal{X}} g^3(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left|q_g^{(2)}(\alpha, x)\right| \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}^2 (q(\alpha, x)) \\
&\quad \cdot \frac{1}{4} \left(h_q^4 \int t_1^6 k^2(t_1) dt_1 + O\left(h_q^5\right)\right) \\
&= O\left(h_q^4\right) \\
&= o(1),
\end{aligned}$$

where amongst others, the conditions that

$$\begin{aligned}
\sup_{x \in \mathcal{X}} g(x) &< \infty, \\
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1|X_1=x}(q(\alpha, x)) &< \infty, \\
\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left|q_g^{(2)}(\alpha, x)\right| &< \infty
\end{aligned}$$

and

$$\int t_1^{12} k^2(t_1) dt_1 < \infty$$

have been invoked.

Combining results, we have for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ that

$$\mathbf{\Sigma}(\alpha, x) = \alpha(1-\alpha) g(x) \begin{bmatrix} \int k^2(t_1) dt_1 & \int t_1 k^2(t_1) dt_1 \\ \int t_1 k^2(t_1) dt_1 & \int t_1^2 k^2(t_1) dt_1 \end{bmatrix} + o(1).$$

The next step in the proof involves the verification of the convergence in (14) by showing that Liapounov's condition holds. In this connection, we note that for Liapounov's condition to hold in this case it is sufficient if both

$$\begin{aligned}
& (nh_q)^{-1} h_q^{-1} E\left[(\psi_\alpha(\bar{u}_1^*(x)) - E[\psi_\alpha(\bar{u}_1^*(x))| X_1])^4 k^4\left(h_q^{-1}(X_1 - x)\right) \tau_n^4(g(X_1))\right] \\
\to \quad & 0 \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
& (nh_q)^{-1} h_q^{-1} E\left[(\psi_\alpha(\bar{u}_1^*(x)) - E[\psi_\alpha(\bar{u}_1^*(x))| X_1])^4 k^4\left(h_q^{-1}(X_1 - x)\right) \tau_n^4(g(X_1)) \right. \\
& \left. \cdot \left[h_q^{-1}(X_1 - x)\right]^8\right] \\
\to \quad & 0 \qquad (18)
\end{aligned}$$

hold for any $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$.

Conditions (17) and (18) are verified in sequence.

With respect to (17), let $T_1 \equiv h_q^{-1}(X_1 - x)$. Let ξ_1 be a point on the line segment between x and $X_1 = x + h_q T_1$. Let $\zeta_1(\alpha, h_q)$ be a point in the interval

$$\left(q(\alpha, X_1) - \frac{h_q^2}{2} g(X_1) T_1^2 q_g^{(2)}(\alpha, \xi_1), q(\alpha, X_1) \right).$$

Define

$$\mu_{h_q} \equiv \frac{1}{2} g(X_1)(X_1 - x)^2 q_g^{(2)}(\alpha, \xi_1) f_1(\zeta_1(\alpha, h_q)).$$

Recall from (15) that $E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \middle| X_1\right] = \mu_{h_q}$, and from (16) that

$$E\left[\psi_\alpha^2(\bar{u}_1^*(x)) \middle| X_1\right] = \alpha(1-\alpha) + (2\alpha - 1)\mu_{h_q}.$$

Similarly, it is possible to show that

$$\begin{aligned} E\left[\psi_\alpha^3(\bar{u}_1^*(x)) \middle| X_1\right] &= \alpha(1-\alpha)(2\alpha-1) + \left(3\alpha^2 - 3\alpha + 1\right)\mu_{h_q}, \\ E\left[\psi_\alpha^4(\bar{u}_1^*(x)) \middle| X_1\right] &= \alpha(1-\alpha)\left(3\alpha^2 - 3\alpha + 1\right) + \left(4\alpha^3 - 6\alpha^2 + 4\alpha - 1\right)\mu_{h_q}. \end{aligned}$$

It follows that

$$\begin{aligned} & E\left[\psi_\alpha^4(\bar{u}_1^*(x)) k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & E\left[E\left[\psi_\alpha^4(\bar{u}_1^*(x)) \middle| X_1\right] k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & \alpha(1-\alpha)\left(3\alpha^2 - 3\alpha + 1\right) E\left[k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ & + \left(4\alpha^3 - 6\alpha^2 - 4\alpha - 1\right) E\left[\mu_{h_q} k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right]; \end{aligned}$$

that

$$\begin{aligned} & E\left[\psi_\alpha^3(\bar{u}_1^*(x)) \mu_{h_q} k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & E\left[E\left[\psi_\alpha^3(\bar{u}_1^*(x)) \middle| X_1\right] \mu_{h_q} k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & \alpha(1-\alpha)(2\alpha-1) E\left[\mu_{h_q} k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ & + \left(3\alpha^2 - 3\alpha + 1\right) E\left[\mu_{h_q}^2 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right]; \end{aligned}$$

that

$$\begin{aligned} & E\left[\psi_\alpha^2(\bar{u}_1^*(x)) \mu_{h_q}^2 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & E\left[E\left[\psi_\alpha^2(\bar{u}_1^*(x)) \middle| X_1\right] \mu_{h_q}^2 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & \alpha(1-\alpha) E\left[\mu_{h_q}^2 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ & + (2\alpha - 1) E\left[\mu_{h_q}^3 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right]; \end{aligned}$$

and that

$$\begin{aligned} & E\left[\psi_\alpha(\bar{u}_1^*(x)) \mu_{h_q}^3 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & E\left[E\left[\psi_\alpha(\bar{u}_1^*(x)) \middle| X_1\right] \mu_{h_q}^3 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & E\left[\mu_{h_q}^4 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right]. \end{aligned}$$

Note that

$$\begin{aligned} & h_q^{-1} E\left[\left(\psi_\alpha(\bar{u}_1^*(x)) - E\left[\psi_\alpha(\bar{u}_1^*(x)) \middle| X_1\right]\right)^4 k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right] \\ = \; & h_q^{-1} E\left[\left(\psi_\alpha^4(\bar{u}_1^*(x)) - 4\psi_\alpha^3(\bar{u}_1^*(x))\mu_{h_q} + 6\psi_\alpha^2(\bar{u}_1^*(x))\mu_{h_q}^2 - 4\psi_\alpha(\bar{u}_1^*(x))\mu_{h_q}^3 + \mu_{h_q}^4\right) \right. \\ & \left. \cdot k^4\left(h_q^{-1}(X_1 - x)\right)\tau_n^4(g(X_1))\right]. \end{aligned}$$

As such, we consider the following asymptotic representations in sequence, where the various conditions imposed on $k(\cdot)$, h_q, and on $g(x)$, $f_{Y_1|X_1=x}(q(\alpha, x))$ and $q^{(2)}(\alpha, x)$ for $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$, amongst others, are invoked:

(1)

$$
\begin{aligned}
& h_q^{-1} E\left[k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & h_q^{-1} \int k^4\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n^4\left(g\left(x_1\right)\right) g\left(x_1\right) d x_1 \\
= & \int k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) g\left(x+h_q t_1\right) d t_1 \\
= & \tau_n^4(g(x)) g(x) \int k^4\left(t_1\right) d t_1+O\left(h_q\right) \\
= & g(x) \int k^4\left(t_1\right) d t_1+\left(\tau_n^4(g(x))-1\right) g(x) \int k^4\left(t_1\right) d t_1+O\left(h_q\right) \\
= & g(x) \int k^4\left(t_1\right) d t_1+O\left(n^{-\zeta}+h_q\right) \\
= & g(x) \int k^4\left(t_1\right) d t_1+o(1) .
\end{aligned}
$$

(2) For fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some $(\gamma_1, \gamma_2) \in (0,1)^2$, we have the following:

a)

$$
\begin{aligned}
& h_q^{-1} E\left[\mu_{h_q} k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{2} h_q^{-1} E\left[g\left(X_1\right)\left(X_1-x\right)^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right)\right. \\
& \left.\cdot k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{2} h_q^{-1} \int g^2\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right) \\
& \cdot f_1\left(q\left(\alpha, x_1\right)-\gamma_2 \cdot \frac{1}{2} g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right) \\
& \cdot k^4\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n^4\left(g\left(x_1\right)\right) d x_1 \\
= & \frac{h_q^2}{2} \int g^2\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right) \\
& \cdot f_1\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= & \frac{1}{2}\left(h_q^2 \int g^2\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right) f_1\left(q\left(\alpha, x+h_q t_1\right)\right)\right. \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
& \left.+O\left(h_q^4\right)\right) \\
= & \frac{1}{2}\left(h_q^2 g^2(x) q_g^{(2)}(\alpha, x) f_{Y_1 \mid X_1=x}(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^2 k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^3\right)\right) \\
\leq & \sup_{x \in \mathcal{X}} g^2(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}}\left|q_g^{(2)}(\alpha, x)\right| \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}(q(\alpha, x)) \\
& \cdot \frac{h_q^2}{2}\left(\int t_1^2 k^4\left(t_1\right) d t_1+O\left(h_q^3\right)\right) \\
= & O\left(h_q^2\right) \\
= & o(1),
\end{aligned}
$$

b)

$$
\begin{aligned}
& h_q^{-1} E\left[\mu_{h_q}^2 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{4} h_q^{-1} E\left[g^2\left(X_1\right)\left(X_1-x\right)^4\left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^2 f_1^2\left(\zeta_1\left(\alpha, h_q\right)\right)\right. \\
& \left.\cdot k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{4} h_q^{-1} \int g^3\left(x_1\right)\left(x_1-x\right)^4\left(q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right)^2 \\
& \cdot f_1^2\left(q\left(\alpha, x_1\right)-\gamma_2 \cdot \frac{1}{2} g\left(x_1\right)(\boldsymbol{x}_1-\boldsymbol{x})^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right) \\
& \cdot k^4\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n^4\left(g\left(x_1\right)\right) d x_1 \\
= & \frac{h_q^4}{4} \int g^3\left(x+h_q t_1\right) t_1^4\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^2 \\
& \cdot f_1^2\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= & \frac{1}{4}\left(h_q^4 \int g^3\left(x+h_q t_1\right) t_1^4\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^2 f_1^2\left(q\left(\alpha, x+h_q t_1\right)\right)\right. \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
& \left.+O\left(h_q^6\right)\right) \\
= & \frac{1}{4}\left(h_q^4 g^3(x)\left(q_g^{(2)}(\alpha, x)\right)^2 f_{Y_1 \mid X_1=x}^2(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^4 k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^5\right)\right) \\
\leq & \sup_{x \in \mathcal{X}} g^3(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}}\left(q_g^{(2)}(\alpha, x)\right)^2 \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}^2(q(\alpha, x)) \\
& \cdot \frac{1}{4}\left(h_q^4 \int t_1^4 k^4\left(t_1\right) d t_1+O\left(h_q^5\right)\right) \\
= & O\left(h_q^4\right) \\
= & o(1),
\end{aligned}
$$

c)

$$
\begin{aligned}
& h_q^{-1} E\left[\mu_{h_q}^3 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{8} h_q^{-1} E\left[g^3\left(X_1\right)\left(X_1-x\right)^6\left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^3 f_1^3\left(\zeta_1\left(\alpha, h_q\right)\right)\right. \\
& \left.\cdot k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{8} h_q^{-1} \int g^4\left(x_1\right)\left(x_1-x\right)^6\left(q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right)^3 \\
& \cdot f_1^3\left(q\left(\alpha, x_1\right)-\gamma_2 \cdot \frac{1}{2} g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right) \\
& \cdot k^4\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n^4\left(g\left(x_1\right)\right) d x_1 \\
= & \frac{h_q^6}{8} \int g^4\left(x+h_q t_1\right) t_1^6\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^3 \\
& \cdot f_1^3\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= & \frac{1}{8}\left(h_q^6 \int g^4\left(x+h_q t_1\right) t_1^6\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^3 f_1^3\left(q\left(\alpha, x+h_q t_1\right)\right)\right. \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
& \left.+O\left(h_q^8\right)\right) \\
= & \frac{1}{8}\left(h_q^6 g^4(x)\left(q_g^{(2)}(\alpha, x)\right)^3 f_{Y_1 \mid X_1=x}^3(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^6 k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^7\right)\right) \\
\leq & \sup_{x \in \mathcal{X}} g^4(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}}\left|q_g^{(2)}(\alpha, x)\right|^3 \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}^3(q(\alpha, x)) \\
& \cdot \frac{1}{8}\left(h_q^6 \int t_1^6 k^4\left(t_1\right) d t_1+O\left(h_q^7\right)\right) \\
= & O\left(h_q^6\right) \\
= & o(1),
\end{aligned}
$$

d)

$$
\begin{aligned}
& h_q^{-1} E\left[\mu_{h_q}^4 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{16} h_q^{-1} E\left[g^4\left(X_1\right)\left(X_1-x\right)^8\left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^4 f_1^4\left(\zeta_1\left(\alpha, h_q\right)\right)\right. \\
& \left.\cdot k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{16} h_q^{-1} \int g^5\left(x_1\right)\left(x_1-x\right)^8\left(q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right)^4 \\
& \cdot f_1^4\left(q\left(\alpha, x_1\right)-\gamma_2 \cdot \frac{1}{2} g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right) \\
& \cdot k^4\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n^4\left(g\left(x_1\right)\right) d x_1 \\
= & \frac{h_q^8}{16} \int g^5\left(x+h_q t_1\right) t_1^8\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^4 \\
& \cdot f_1^4\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= & \frac{1}{16}\left(h_q^8 \int g^5\left(x+h_q t_1\right) t_1^8\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^4 f_1^4\left(q\left(\alpha, x+h_q t_1\right)\right)\right. \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
& \left.+O\left(h_q^{10}\right)\right) \\
= & \frac{1}{16}\left(h_q^8 g^5(x)\left(q_g^{(2)}(\alpha, x)\right)^4 f_{Y_1 \mid X_1=x}^4(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^8 k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^9\right)\right) \\
\leq & \sup _{x \in \mathcal{X}} g^5(x) \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}}\left(q_g^{(2)}(\alpha, x)\right)^4 \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}^4(q(\alpha, x)) \\
& \cdot \frac{1}{16}\left(h_q^8 \int t_1^8 k^4\left(t_1\right) d t_1+O\left(h_q^9\right)\right) \\
= & O\left(h_q^8\right) \\
= & o(1) .
\end{aligned}
$$

Combining results, we have for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ that

$$
\begin{aligned}
& h_q^{-1} E\left[\left(\psi_\alpha\left(\bar{u}_1^*(x)\right)-E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \mid X_1\right]\right)^4 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
\rightarrow \quad & \alpha(1-\alpha)\left(3 \alpha^2-3 \alpha+1\right) g(x) \int k^4\left(t_1\right) d t_1 .
\end{aligned}
$$

As such, provided that $n h_q \rightarrow \infty$, (17) holds.

Now consider (18). We have

$$
\begin{aligned}
& h_q^{-1} E\left[\left(\psi_\alpha\left(\bar{u}_1^*(x)\right)-E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \mid X_1\right]\right)^4 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right) \cdot\left[h_q^{-1}\left(X_1-x\right)\right]^8\right] \\
= \quad & h_q^{-1} E\left[\left(\psi_\alpha^4\left(\bar{u}_1^*(x)\right)-4 \psi_\alpha^3\left(\bar{u}_1^*(x)\right) \mu_{h_q}+6 \psi_\alpha^2\left(\bar{u}_1^*(x)\right) \mu_{h_q}^2-4 \psi_\alpha\left(\bar{u}_1^*(x)\right) \mu_{h_q}^3+\mu_{h_q}^4\right)\right. \\
& \left.\cdot k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right) \cdot\left[h_q^{-1}\left(X_1-x\right)\right]^8\right] .
\end{aligned}
$$

As such, we consider the following asymptotic representations in sequence, where the various conditions imposed on $k(\cdot)$, h_q, and on $g(x)$, $f_{Y_1 \mid X_1=x}(q(\alpha, x))$ and $q^{(2)}(\alpha, x)$ for $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$, amongst others, are invoked:

(1)

$$
\begin{aligned}
& h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \int t_1^8 k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) g\left(x+h_q t_1\right) d t_1 \\
= & \tau_n^4(g(x)) g(x) \int t_1^8 k^4\left(t_1\right) d t_1+O\left(h_q\right) \\
= & g(x) \int t_1^8 k^4\left(t_1\right) d t_1+\left(\tau_n^4(g(x))-1\right) g(x) \int t_1^8 k^4\left(t_1\right) d t_1+O\left(h_q\right) \\
= & g(x) \int t_1^8 k^4\left(t_1\right) d t_1+O\left(n^{-\zeta}+h_q\right) \\
= & g(x) \int t_1^8 k^4\left(t_1\right) d t_1+o(1).
\end{aligned}
$$

(2) For fixed $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ and some $(\gamma_1, \gamma_2) \in (0,1)^2$, we have the following:

a)

$$
\begin{aligned}
& h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8 \mu_{h_q} k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{2} h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^{10} g\left(X_1\right) q_g^{(2)}\left(\alpha, \xi_1\right)\right. \\
& \left.\cdot f_1\left(\zeta_1\left(\alpha, h_q\right)\right) k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{h_q^2}{2} \int t_1^{10} g^2\left(x+h_q t_1\right) q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right) \\
& \cdot f_1\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= & \frac{1}{2}\left(h_q^2 g^2(x) q_g^{(2)}(\alpha, x) f_{Y_1 \mid X_1=x}(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^8 k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^3\right)\right) \\
\leq & \sup_{x \in \mathcal{X}} g^2(x) \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}}\left|q_g^{(2)}(\alpha, x)\right| \cdot \sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}(q(\alpha, x)) \\
& \cdot \frac{1}{2}\left(h_q^2 \int t_1^{10} k^4\left(t_1\right) d t_1+O\left(h_q^3\right)\right) \\
= & O\left(h_q^2\right) \\
= & o(1),
\end{aligned}
$$

b)

$$
\begin{aligned}
& h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8 \mu_{h_q}^2 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= \; & \frac{1}{4} h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^{12} g^2\left(X_1\right)\left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^2\right. \\
& \left.\cdot f_1^2\left(\zeta_1\left(\alpha, h_q\right)\right) k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= \; & \frac{h_q^4}{4} \int t_1^{12} g^3\left(x+h_q t_1\right)\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^2 \\
& \cdot f_1^2\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= \; & \frac{1}{4}\left(h_q^4 g^3(x)\left(q_g^{(2)}(\alpha, x)\right)^2 f_{Y_1 \mid X_1=x}^2(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^{12} k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^3\right)\right) \\
\leq \; & \sup _{x \in \mathcal{X}} g^3(x) \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}}\left(q_g^{(2)}(\alpha, x)\right)^2 \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}^2(q(\alpha, x)) \\
& \cdot \frac{1}{4}\left(h_q^4 \int t_1^{12} k^4\left(t_1\right) d t_1+O\left(h_q^5\right)\right) \\
= \; & O\left(h_q^4\right) \\
= \; & o(1),
\end{aligned}
$$

c)

$$
\begin{aligned}
& h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8 \mu_{h_q}^3 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= \; & \frac{1}{8} h_q^{-1} E\left[g^3\left(X_1\right)\left(X_1-x\right)^6\left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^3\right. \\
& \left.\cdot f_1^3\left(\zeta_1\left(\alpha, h_q\right)\right) k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\left[h_q^{-1}\left(X_1-x\right)\right]^8\right] \\
= \; & \frac{h_q^6}{8} \int t_1^{14} g^4\left(x+h_q t_1\right)\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^3 \\
& \cdot f_1^3\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= \; & \frac{1}{8}\left(h_q^6 g^4(x)\left(q_g^{(2)}(\alpha, x)\right)^3 f_{Y_1 \mid X_1=x}^3(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^{14} k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^7\right)\right) \\
\leq \; & \sup _{x \in \mathcal{X}} g^4(x) \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}}\left|q_g^{(2)}(\alpha, x)\right|^3 \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}^3(q(\alpha, x)) \\
& \cdot \frac{1}{8}\left(h_q^6 \int t_1^{14} k^4\left(t_1\right) d t_1+O\left(h_q^7\right)\right) \\
= \; & O\left(h_q^6\right) \\
= \; & o(1),
\end{aligned}
$$

d)

$$
\begin{aligned}
& h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8 \mu_{h_q}^4 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{1}{16} h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8 g^4\left(X_1\right)\left(X_1-x\right)^8\left(q_g^{(2)}\left(\alpha, \xi_1\right)\right)^4\right. \\
& \left.\cdot f_1^4\left(\zeta_1\left(\alpha, h_q\right)\right) k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{h_q^8}{16} \int t_1^{16} g^5\left(x+h_q t_1\right)\left(q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right)^4 \\
& \cdot f_1^4\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k^4\left(t_1\right) \tau_n^4\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= & \frac{1}{16}\left(h_q^8 g^5(x)\left(q_g^{(2)}(\alpha, x)\right)^4 f_{Y_1 \mid X_1=x}^4(q(\alpha, x)) \tau_n^4(g(x)) \int t_1^{16} k^4\left(t_1\right) d t_1\right. \\
& \left.+O\left(h_q^9\right)\right) \\
\leq & \sup _{x \in \mathcal{X}} g^5(x) \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}}\left(q_g^{(2)}(\alpha, x)\right)^4 \cdot \sup _{\alpha \in \mathcal{A}} \sup _{x \in \mathcal{X}} f_{Y_1 \mid X_1=x}^4(q(\alpha, x)) \\
& \cdot \frac{1}{16}\left(h_q^8 \int t_1^{16} k^4\left(t_1\right) d t_1+O\left(h_q^9\right)\right) \\
= & O\left(h_q^8\right) \\
= & o(1) .
\end{aligned}
$$

Combining results, we have for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ that

$$
\begin{aligned}
& h_q^{-1} E\left[\left[h_q^{-1}\left(X_1-x\right)\right]^8\left(\psi_\alpha\left(\bar{u}_1^*(x)\right)-E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \mid X_1\right]\right)^4 k^4\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n^4\left(g\left(X_1\right)\right)\right] \\
\rightarrow\; & \alpha(1-\alpha)\left(3 \alpha^2-3 \alpha+1\right) g(x) \int t_1^8 k^4\left(t_1\right) d t_1 .
\end{aligned}
$$

As such, provided that $n h_q \rightarrow \infty$, (18) holds.

It follows that Liapounov's condition holds and the convergence in (14) holds for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$.

2.3 Second step: Asymptotic unbiasedness of $q_n^*(\alpha, x)$

Consider

$$
E\left[\left[\begin{array}{c}
\psi_\alpha\left(\bar{u}_1^*(x)\right) k\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n\left(g\left(X_1\right)\right) \\
h_q^{-1}\left(X_1-x\right) \psi_\alpha\left(\bar{u}_1^*(x)\right) k\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n\left(g\left(X_1\right)\right)
\end{array}\right]\right] .
$$

As before, let $T_1 \equiv h_q^{-1}\left(X_1-x\right)$, so $X_1=x+h_q T_1$. Let ξ_1 be a point on the line segment between x and X_1. Let $\zeta_1\left(\alpha, h_q\right)$ be a point in the interval

$$
\left(q\left(\alpha, X_1\right)-\frac{1}{2} g\left(X_1\right)\left(X_1-x\right)^2 q_g^{(2)}\left(\alpha, \xi_1\right), q\left(\alpha, X_1\right)\right) .
$$

Let $(\gamma_1, \gamma_2) \in (0,1)^2$ be suitable constants. Assume that $g(\cdot)$ is at least $(L-1)$-times continuously differentiable on $\mathcal{X}$. We have the following:

$$\begin{aligned}
& E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) k\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n\left(g\left(X_1\right)\right)\right] \\
= \; & E\left[E\left[\psi_\alpha\left(\bar{u}_1^*(x)\right) \mid X_1\right] k\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n\left(g\left(X_1\right)\right)\right] \\
= \; & \frac{1}{2} E\left[g\left(X_1\right)\left(X_1-x\right)^2 q_g^{(2)}\left(\alpha, \xi_1\right) f_1\left(\zeta_1\left(\alpha, h_q\right)\right) k\left(h_q^{-1}\left(X_1-x\right)\right) \tau_n\left(g\left(X_1\right)\right)\right] \\
= \; & \frac{1}{2} \int g^2\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right) \\
& \cdot f_1\left(q\left(\alpha, x_1\right)-\gamma_2 \cdot \frac{1}{2} g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha, \gamma_1\left(x_1-x\right)+x\right)\right) \\
& \cdot k\left(h_q^{-1}\left(x_1-x\right)\right) \tau_n(g(x)) d x_1 \\
= \; & \frac{1}{2} h_q^3 \int g^2\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right) \\
& \cdot f_1\left(q\left(\alpha, x+h_q t_1\right)-\gamma_2 \cdot \frac{h_q^2}{2} g\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right)\right) \\
& \cdot k\left(t_1\right) \tau_n\left(g\left(x+h_q t_1\right)\right) d t_1 \\
= \; & \frac{1}{2} h_q^3\left(\int g^2\left(x+h_q t_1\right) t_1^2 q_g^{(2)}\left(\alpha, \gamma_1 h_q t_1+x\right) f_1\left(q\left(\alpha, x+h_q t_1\right)\right) k\left(t_1\right) \tau_n\left(g\left(x+h_q t_1\right)\right) d t_1\right. \\
& \left.+O\left(h_q^4\right)\right) \\
= \; & \frac{1}{2} h_q\left(f_{Y_1 \mid X_1=x}\left(q(\alpha, x)\right) \tau_n(g(x)) q_g^{(2)}(\alpha, x) \cdot h_q^2 \int t_1^2 k\left(t_1\right) g^2\left(x+h_q t_1\right) d t_1\right. \\
& \left.+O\left(h_q^3\right)\right) \\
= \; & \frac{1}{2} h_q\left\{f_{Y_1 \mid X_1=x}\left(q(\alpha, x)\right) \tau_n(g(x)) q_g^{(2)}(\alpha, x)\left[\frac{D_x^{L-2} g^2(x)}{(L-2)!} \cdot h_q^L \int t_1^L k\left(t_1\right) d t_1\right.\right. \\
& \left.+\int_0^1(1-u)^{L-2} \int D_x^{L-1} g^2\left(x+u h_q t_1\right) t_1^2 k\left(t_1\right) d t_1 d u\right] \\
& \left.+O\left(h_q^3\right)\right\} \\
= \; & \frac{1}{2} h_q\left[f_{Y_1 \mid X_1=x}\left(q(\alpha, x)\right) q_g^{(2)}(\alpha, x) \frac{D_x^{L-2} g^2(x)}{(L-2)!} \cdot h_q^L \int t_1^L k\left(t_1\right) d t_1\right. \\
& +O\left(n^{-\zeta} h_q^L\right) \\
& \left.+O\left(h_q^{L+1}\right)+O\left(h_q^3\right)\right] \\
= \; & \frac{1}{2} h_q f_{Y_1 \mid X_1=x}\left(q(\alpha, x)\right) q_g^{(2)}(\alpha, x) \frac{D_x^{L-2} g^2(x)}{(L-2)!} \cdot h_q^L \int t_1^L k\left(t_1\right) d t_1 \\
& +O\left(\left(n^{-\zeta}+h_q\right) h_q^{L+1}\right) \\
= \; & \frac{1}{2} f_{Y_1 \mid X_1=x}\left(q(\alpha, x)\right) q_g^{(2)}(\alpha, x) \frac{D_x^{L-2} g^2(x)}{(L-2)!} \cdot h_q^{L+1} \int t_1^L k\left(t_1\right) d t_1 \\
& +o\left(h_q^{L+1}\right) .
\end{aligned} \qquad (19)$$

Similarly, if $g(\cdot)$ is at least $(L-2)$-times continuously differentiable on $\mathcal{X}$, the following holds:

$$\begin{aligned}
& E\left[h_q^{-1}\left(X_1-x\right)\psi_\alpha\left(\bar{u}_1^*(x)\right)k\left(h_q^{-1}\left(X_1-x\right)\right)\tau_n\left(g\left(X_1\right)\right)\right] \\
= \; & \frac{1}{2}\int h_q^{-1}\left(x_1-x\right)^3 g^2\left(x_1\right)q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right) \\
& \cdot f_1\left(q\left(\alpha,x_1\right)-\gamma_2\cdot\frac{1}{2}g\left(x_1\right)\left(x_1-x\right)^2 q_g^{(2)}\left(\alpha,\gamma_1\left(x_1-x\right)+x\right)\right) \\
& \cdot k\left(h_q^{-1}\left(x_1-x\right)\right)\tau_n\left(g\left(x_1\right)\right)dx_1 \\
= \; & \frac{1}{2}h_q\left(h_q^2\int t_1^3 g^2\left(x+h_q t_1\right)q_g^{(2)}\left(\alpha,\gamma_1 h_q t_1+x\right)\right. \\
& \cdot f_1\left(q\left(\alpha,x+h_q t_1\right)\right)k\left(t_1\right)\tau_n\left(g\left(x+h_q t_1\right)\right)dt_1 \\
& \left.+O\left(h_q^4\right)\right) \\
= \; & \frac{1}{2}\left\{f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\tau_n\left(g(x)\right)\right. \\
& \cdot\left[\frac{D_x^{L-3}g^2(x)}{(L-3)!}\cdot q_g^{(2)}(\alpha,x)\cdot h_q^{L-1}\int t_1^L k\left(t_1\right)dt_1\right. \\
& \left.+h_q^2\int_0^1(1-u)^{L-3}\int D_x^{L-2}q_g^{(2)}(\alpha,x)g^2\left(x+uh_q t_1\right)t_1^3 k\left(t_1\right)dt_1 du\right] \\
& \left.+O\left(h_q^3\right)\right\} \\
= \; & \frac{1}{2}h_q\left[f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\frac{D_x^{L-3}g^2(x)}{(L-3)!}q_g^{(2)}(\alpha,x)h_q^{L-1}\int t_1^L k\left(t_1\right)dt_1\right. \\
& \left.+O\left(n^{-\zeta}h_q^{L-1}\right)+O\left(h_q^L\right)+O\left(h_q^3\right)\right] \\
= \; & \frac{1}{2}h_q f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\frac{D_x^{L-3}g^2(x)}{(L-3)!}q_g^{(2)}(\alpha,x)\cdot h_q^{L-1}\int t_1^L k\left(t_1\right)dt_1 \\
& +O\left(\left(n^{-\zeta}+h_q\right)h_q^L\right) \\
= \; & \frac{1}{2}h_q^L f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\frac{D_x^{L-3}g^2(x)}{(L-3)!}q_g^{(2)}(\alpha,x)\int t_1^L k\left(t_1\right)dt_1 \\
& +o\left(h_q^L\right).
\end{aligned} \tag{20}$$

Combine (19) with (20) to deduce that

$$\begin{aligned}
& \sqrt{\frac{n}{h_q}}E\left[\left[\begin{array}{c}\psi_\alpha\left(\bar{u}_1^*(x)\right)k\left(h_q^{-1}\left(X_1-x\right)\right)\tau_n\left(g\left(X_1\right)\right) \\ h_q^{-1}\left(X_1-x\right)\psi_\alpha\left(\bar{u}_1^*(x)\right)k\left(h_q^{-1}\left(X_1-x\right)\right)\tau_n\left(g\left(X_1\right)\right)\end{array}\right]\right] \\
\approx \; & \frac{1}{2}\sqrt{nh_q}f_{Y_1|X_1=x}\left(q(\alpha,x)\right)\left[\begin{array}{c}h_q^L\cdot q_g^{(2)}(\alpha,x)\frac{D_x^{L-2}g^2(x)}{(L-2)!}\int t_1^L k\left(t_1\right)dt_1 \\ h_q^{L-1}\cdot q_g^{(2)}(\alpha,x)\frac{D_x^{L-3}g^2(x)}{(L-3)!}\int t_1^L k\left(t_1\right)dt_1\end{array}\right] \\
= \; & o(1),
\end{aligned}$$

where the condition

$$\sqrt{nh_q}\cdot h_q^{L-1}=\sqrt{n}h_q^{L-\frac{1}{2}}\to 0.$$

has been invoked.

2.4 Conclusion

Combine the results of the two steps above with (13) to deduce that for each $\boldsymbol{\delta}\in\mathbb{R}^2$ and each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$,

$$Z_{n1}(\boldsymbol{\delta})\xrightarrow{d}-\left[\begin{array}{cc}\delta_0, & \delta_1\end{array}\right]\bar{\boldsymbol{W}}(\alpha,x),$$

where

$$\bar{W}(\alpha,x) \sim N\left(\mathbf{0},\ \alpha(1-\alpha)g\left(x\right)\left[\begin{array}{cc} \int k^2\left(t_1\right)dt_1 & \int t_1 k^2\left(t_1\right)dt_1 \\ \int t_1 k^2\left(t_1\right)dt_1 & \int t_1^2 k^2\left(t_1\right)dt_1 \end{array}\right]\right).$$

Combine this result with (1) and (12) to deduce that

$$\begin{aligned} Z_n(\boldsymbol{\delta}) \quad &\stackrel{d}{\to} \quad -\left[\begin{array}{cc}\delta_0 & \delta_1\end{array}\right]\bar{\boldsymbol{W}}(\alpha,x)+\frac{1}{2}f_{Y_1|X_1=x}\left(q(\alpha,x)\right)g(x) \\ &\qquad \cdot\left[\begin{array}{cc}\delta_0 & \delta_1\end{array}\right]\left[\begin{array}{cc} \int k^2\left(t_1\right)dt_1 & \int t_1 k^2\left(t_1\right)dt_1 \\ \int t_1 k^2\left(t_1\right)dt_1 & \int t_1^2 k^2\left(t_1\right)dt_1 \end{array}\right]\left[\begin{array}{c}\delta_0\\ \delta_1\end{array}\right] \\ &\equiv \quad Z_\infty(\boldsymbol{\delta}) \end{aligned}$$

for each $\boldsymbol{\delta}\in\mathbb{R}^2$ and each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$.

Note that the convexity of $Z_\infty(\boldsymbol{\delta})$ implies the uniqueness of its minimiser. In addition, by the "$\arg\min$ continuous mapping theorem" (e.g., Pollard 1991, Hjørt and Pollard 1993, Knight 1998) we have

$$\arg\min Z_n(\boldsymbol{\delta}) \stackrel{d}{\to} \arg\min Z_\infty(\boldsymbol{\delta}) \tag{21}$$

for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$.

The convergence in (21) implies that for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$,

$$\begin{aligned} \left[\begin{array}{c}\hat{\delta}_{n0}(\alpha,x)\\ \hat{\delta}_{n1}(\alpha,x)\end{array}\right] \quad &= \quad \sqrt{nh_q}\left[\begin{array}{c} \hat{q}_n^*(\alpha,x)-\frac{q(\alpha,x)}{g(x)} \\ h_q\left(\hat{q}_{n1}^*(\alpha,x)-q_g^{(1)}(\alpha,x)\right)\end{array}\right] \\ &\stackrel{d}{\to} \quad f_{Y_1|X_1=x}^{-1}\left(q(\alpha,x)\right)g^{-1}(x) \\ &\qquad \cdot\left[\begin{array}{cc} \int k^2\left(t_1\right)dt_1 & \int t_1 k^2\left(t_1\right)dt_1 \\ \int t_1 k^2\left(t_1\right)dt_1 & \int t_1^2 k^2\left(t_1\right)dt_1 \end{array}\right]^{-1}\bar{\boldsymbol{W}}(\alpha,x). \end{aligned}$$

Proposition 3.1 is immediate.

3 Proofs of Theorems 3.2 and 3.3

3.1 Preliminaries

The proofs of Theorems 3.2 and 3.3 involve the use of the following conceptual device.

Let $\tilde{\tau}_n(\cdot)$ denote a trimming function satisfying all the conditions of part (1) of Assumption A.5. In particular, define

$$\tilde{\tau}_n(u)=\begin{cases} 1 & , \quad u\geq 2n^{-\zeta_1} \\ 0 & , \quad u\leq n^{-\zeta_1} \\ \bar{\tau}_{n1}(u) & , \quad u\in\left(n^{-\zeta_1},2n^{-\zeta_1}\right)\end{cases}$$

where $\zeta_1>0$ is some constant, and where $\bar{\tau}_{n1}(u)$ is a twice differentiable distribution function with the form $\bar{\tau}_{n1}(u)=\int_{-\infty}^{u}n^{\zeta_1}\sigma\left(n^{\zeta_1}t-1\right)dt$, where $\sigma(\cdot)$ is a differentiable density function uniformly bounded and supported on $[0,1]$ with $\sigma(0)=\sigma(1)=0$, and with $0<\left|\sigma^{(1)}(0)\right|<\left|\sigma^{(1)}(1)\right|<\infty$.

It should be emphasised that $\tilde{\tau}_n(\cdot)$ and the associated trimming parameter ζ_1 are purely conceptual—their role is simply to enable calculations involving the second and fourth moments of the empirically infeasible quantity $\bar{\theta}_n(\alpha,x)$, which is given as follows:

$$\bar{\theta}_n(\alpha,x)\equiv\frac{1}{n}\sum_{i=1}^{n}D_x^{\nu}K_h\left(X_i-x\right)\frac{q\left(\alpha,X_i\right)}{g\left(X_i\right)}\tilde{\tau}_n\left(g\left(X_i\right)\right). \tag{22}$$

In what follows—particularly in the proof of Theorem 3.3 below—it is assumed that the trimming parameter ζ_1 is set so as to make any residual trimming effects associated with the moments of $\bar{\theta}_n(\alpha,x)$ asymptotically negligible.

3.2 Proof of Theorem 3.2

Begin by considering $\left|\hat{\theta}_n(\alpha,x)-\bar{\theta}_n(\alpha,x)\right|$. We have

$$
\begin{aligned}
& \left|\hat{\theta}_n(\alpha,x)-\bar{\theta}_n(\alpha,x)\right| \\
\leq\; & \frac{1}{n}\sum_{i=1}^{n}\left(\sup_{x'\in\mathcal{X}}\left|D_{x'}^{\nu}K_h\left(X_i-x'\right)\right|\right)\cdot\left|\hat{q}_n^{*}\left(\alpha,X_i\right)-\frac{q\left(\alpha,X_i\right)}{g\left(X_i\right)}\tilde{\tau}_n\left(g\left(X_i\right)\right)\right| \\
\leq\; & \frac{1}{nh}\sum_{i=1}^{n}\left(\sup_{x'\in\mathcal{X}}\left|D_{x'}^{\nu}K\left(h^{-1}\left(X_i-x'\right)\right)\right|\right) \\
& \cdot\left|\hat{q}_n^{*}\left(\alpha,x+hT_i\right)-\frac{q\left(\alpha,x+hT_i\right)}{g\left(x+hT_i\right)}\tilde{\tau}_n\left(g\left(x+hT_i\right)\right)\right| \\
=\; & \frac{1}{nh}\sum_{i=1}^{n}\left(\sup_{x'\in\mathcal{X}}\left|D_{x'}^{\nu}K\left(h^{-1}\left(X_i-x'\right)\right)\right|\right) \\
& \cdot\left|\frac{1}{\sqrt{nh_q}}\cdot\sqrt{nh_q}\left(\hat{q}_n^{*}\left(\alpha,x+hT_i\right)-\frac{q\left(\alpha,x+hT_i\right)}{g\left(x+hT_i\right)}\right)\right. \\
& \left.+\frac{q\left(\alpha,x+hT_i\right)}{g\left(x+hT_i\right)}-\frac{q\left(\alpha,x+hT_i\right)}{g\left(x+hT_i\right)}\tilde{\tau}_n\left(g\left(x+hT_i\right)\right)\right| \\
=\; & O_p\left(\frac{1}{\sqrt{nh_q}}\right)+O\left(n^{-\zeta_1}\right),
\end{aligned}
\tag{23}
$$

where the result of Proposition 3.1 that $\sqrt{nh_q}\left(q_n^{*}\left(\alpha,x\right)-q(\alpha,x)/g(x)\right)=O_p(1)$ for each $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$ has been exploited, along with the various assumptions on $g(\cdot)$, $K(\cdot)$, $q(\cdot,\cdot)$ and h.

Next, note that

$$
\begin{aligned}
& E\left[\bar{\theta}_n(\alpha,x)\right] \\
=\; & D_x^{\nu}\int h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)\tilde{\tau}_n\left(g\left(x_1\right)\right)dx_1 \\
=\; & D_x^{\nu}\int_{x_1:\,g(x_1)\geq 2n^{-\zeta_1}}h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)dx_1 \\
& +D_x^{\nu}\int_{x_1:\,n^{-\zeta_1}<g(x_1)<2n^{-\zeta_1}}h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)\tilde{\tau}_n\left(g\left(x_1\right)\right)dx_1 \\
=\; & D_x^{\nu}\int h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)dx_1-D_x^{\nu}\int_{g(x_1)<2n^{-\zeta_1}}h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)dx_1 \\
& +D_x^{\nu}\int_{n^{-\zeta_1}<g(x_1)<2n^{-\zeta_1}}\int h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)\tilde{\tau}_n\left(g\left(x_1\right)\right)dx_1 \\
=\; & D_x^{\nu}\int h^{-1}K\left(h^{-1}\left(x_1-x\right)\right)q\left(\alpha,x_1\right)dx_1+O\left(n^{-\zeta_1}\right) \\
=\; & D_x^{\nu}\int K\left(t_1\right)q\left(\alpha,x+ht_1\right)dt_1+O\left(n^{-\zeta_1}\right) \\
=\; & D_x^{\nu}\int K\left(t_1\right)\left[\sum_{\tau=0}^{5}\frac{D_x^{\tau}q(\alpha,x)}{\tau!}\left(ht_1\right)^{\tau}+\frac{6}{6!}\int_0^1\left(1-u\right)^5D_x^6q\left(\alpha,x+uht_1\right)du\right]dt_1 \\
& +O\left(n^{-\zeta_1}\right) \\
=\; & D_x^{\nu}q(\alpha,x)+\frac{h^2}{2}q^{(2)}(\alpha,x)D_x^{\nu}\int t_1^2K\left(t_1\right)dt_1+\frac{h^4}{4!}D_x^{\nu+4}q(\alpha,x)\int t_1^4K\left(t_1\right)dt_1 \\
& +O\left(h^6\right)+O\left(n^{-\zeta_1}\right).
\end{aligned}
\tag{24}
$$

Invoking the condition that $\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}|D_x^{\tau}q(\alpha,x)|<\infty$ for all $\tau\in\{2,3,\ldots,6\}$, we have

$$\left|E\left[\bar{\theta}_n(\alpha,x)\right]-D_x^{\nu}q(\alpha,x)\right|=O\left(h^2+n^{-\zeta_1}\right). \tag{25}$$

In addition, note that

$$\begin{aligned}
& E\left[\bar{\theta}_n^2(\alpha,x)\right] \\
= \; & E\left[\frac{1}{n^2}\sum_{i,j}D_x^{\nu}K_h\left(X_i-x\right)\frac{q\left(\alpha,X_i\right)}{g\left(X_i\right)}\tilde{\tau}_n\left(g\left(X_i\right)\right)\cdot D_x^{\nu}K_h\left(X_j-x\right)\frac{q\left(\alpha,X_j\right)}{g\left(X_j\right)}\tilde{\tau}_n\left(g\left(X_j\right)\right)\right] \\
= \; & \frac{1}{n}E\left[\left(D_x^{\nu}K_h\left(X_1-x\right)\right)^2\frac{q^2\left(\alpha,X_1\right)}{g^2\left(X_1\right)}\tilde{\tau}_n^2\left(g\left(X_1\right)\right)\right] \\
& +\frac{n(n-1)}{n^2}\left(E\left[D_x^{\nu}K_h\left(X_1-x\right)\frac{q\left(\alpha,X_1\right)}{g\left(X_1\right)}\tilde{\tau}_n\left(g\left(X_1\right)\right)\right]\right)^2.
\end{aligned}$$

Note also that

$$\begin{aligned}
& E\left[\left(D_x^{\nu}K_h\left(X_1-x\right)\right)^2\frac{q^2\left(\alpha,X_1\right)}{g^2\left(X_1\right)}\tilde{\tau}_n^2\left(g\left(X_1\right)\right)\right] \\
= \; & \int h^{-2}\left(D_x^{\nu}K\left(h^{-1}\left(x_1-x\right)\right)\right)^2\frac{q^2\left(\alpha,x_1\right)}{g\left(x_1\right)}\tilde{\tau}_n^2\left(g\left(x_1\right)\right)dx_1 \\
= \; & h^{-1-2\nu}\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2\frac{q^2\left(\alpha,x+ht_1\right)}{g\left(x+ht_1\right)}\tilde{\tau}_n^2\left(g\left(x+ht_1\right)\right)dt_1 \\
= \; & h^{-1-2\nu}\left[\frac{q^2(\alpha,x)}{g(x)}\tilde{\tau}_n^2\left(g(x)\right)\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2dt_1+O\left(h\right)\right] \\
= \; & h^{-1-2\nu}\left[\frac{q^2(\alpha,x)}{g(x)}\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2dt_1+\frac{q^2(\alpha,x)}{g(x)}\left(\tilde{\tau}_n^2\left(g(x)\right)-1\right)\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2dt_1\right. \\
& \left.+O\left(h\right)\right] \\
= \; & h^{-1-2\nu}\left[\frac{q^2(\alpha,x)}{g(x)}\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2dt_1+O\left(n^{-\zeta_1}+h\right)\right]. \qquad (26)
\end{aligned}$$

It follows that

$$\begin{aligned}
E\left[\bar{\theta}_n^2(\alpha,\boldsymbol{x})\right] \;=\; & \frac{1}{nh^{1+2\nu}}\cdot\frac{q^2(\alpha,x)}{g(x)}\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2dt_1+O\left(\frac{n^{-\zeta_1}+h}{nh^{1+2\nu}}\right) \\
& +\left(D_x^{\nu}q(\alpha,x)\right)^2+D_x^{\nu}q(\alpha,x)\cdot D_x^{\nu}q^{(2)}(\alpha,x)\cdot h^2\int t_1^2K\left(t_1\right)dt_1 \\
& +O\left(h^4+n^{-\zeta_1}\right).
\end{aligned}$$

Deduce that

$$\begin{aligned}
& Var\left[\bar{\theta}_n(\alpha,x)\right] \\
= \; & E\left[\left(\bar{\theta}_n-E\left[\bar{\theta}_n(\alpha,x)\right]\right)^2\right] \\
= \; & E\left[\bar{\theta}_n^2(\alpha,x)\right]-\left(E\left[\bar{\theta}_n(\alpha,x)\right]\right)^2 \\
= \; & \left(D_x^{\nu}q(\alpha,x)\right)^2+\frac{1}{nh^{1+2\nu}}\cdot\frac{q^2(\alpha,x)}{g(x)}\int\left(\frac{d^{\nu}}{dt_1^{\nu}}K\left(t_1\right)\right)^2dt_1 \\
& +D_x^{\nu}q(\alpha,x)\cdot D_x^{\nu}q^{(2)}(\alpha,x)\cdot h^2\int t_1^2K\left(t_1\right)dt_1+o\left(\frac{1}{nh^{1+2\nu}}\right) \\
& -\left(D_x^{\nu}q(\alpha,x)\right)^2-D_x^{\nu}q(\alpha,x)\cdot D_x^{\nu}q^{(2)}(\alpha,x)\cdot h^2\int t_1^2K\left(t_1\right)dt_1 \\
& +O\left(h^4+n^{-\zeta_1}\right) \\
= \; & O\left(\frac{1}{nh^{1+2\nu}}+h^4+n^{-\zeta_1}\right).
\end{aligned}$$

Therefore for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$ we have

$$\left|\bar{\theta}_n(\alpha,x) - E\left[\bar{\theta}_n(\alpha,x)\right]\right| = O_p\left(\frac{1}{\sqrt{nh^{1+2\nu}}} + h^2 + n^{-\frac{\zeta_1}{2}}\right), \tag{27}$$

where the conditions that $\sup_{\alpha\in\mathcal{A}}\sup_{x\in\mathcal{X}}\left|q^{(j)}(\alpha,x)\right| < \infty$ for all $j \in \{2,\ldots,6\}$; $\sup_{x\in\mathcal{X}} g(x) < \infty$; and $\int\left(\frac{d^\nu}{dt_1^\nu}K(t_1)\right)^2 dt_1 < \infty$ have been invoked.

Combine (25) and (27) via the triangle inequality to deduce that

$$\left|\bar{\theta}_n(\alpha,x) - D_x^\nu q(\alpha,x)\right| = O_p\left(h^2 + \frac{1}{\sqrt{nh^{1+2\nu}}} + n^{-\frac{\zeta_1}{2}}\right). \tag{28}$$

Invoking the condition that $nh^{1+2\nu} \to \infty$ as $n \to \infty$, combine (23) with (28) to deduce that

$$\left|\hat{\theta}_n(\alpha,x) - D_x^\nu q(\alpha,x)\right| = o_p(1).$$

3.3 Proof of Theorem 3.3

Recall the definition of $\bar{\theta}_n(\alpha,x)$ given above in (22) and begin by showing the asymptotic equivalence of $\hat{\theta}_n(\alpha,x)$ and $\bar{\theta}_n(\alpha,x)$ for each $(\alpha,x) \in \mathcal{A}\times\mathcal{X}$. A similar derivation to (23) above yields

$$\begin{aligned}
& \left|\sqrt{nh^{1+2\nu}}\left(\hat{\theta}_n(\alpha,x) - \bar{\theta}_n(\alpha,x)\right)\right| \\
\leq\ & \frac{1}{nh}\sum_{i=1}^n\left(\sup_{x'\in\mathcal{X}}\left|D_{x'}^\nu K\left(h^{-1}\left(X_i - x'\right)\right)\right|\right) \\
& \cdot\sqrt{nh^{1+2\nu}}\left|\frac{1}{\sqrt{nh_q}}\cdot\sqrt{nh_q}\left(\hat{q}_n^*\left(\alpha, x+hT_i\right) - \frac{q\left(\alpha,x+hT_i\right)}{g\left(x+hT_i\right)}\right)\right| \\
& +\frac{1}{nh}\sum_{i=1}^n\left(\sup_{x'\in\mathcal{X}}\left|D_{x'}^\nu K\left(h^{-1}\left(X_i - x'\right)\right)\right|\right) \\
& \cdot\sqrt{nh^{1+2\nu}}\left|\frac{q\left(\alpha,x+hT_i\right)}{g\left(x+hT_i\right)}\left(1-\tilde{\tau}_n\left(g\left(x+hT_i\right)\right)\right)\right| \\
=\ & o_p(1),
\end{aligned} \tag{29}$$

where the condition that $h^{1+2\nu}/h_q = o(1)$ has been invoked, which in turn implies the convergence $|\hat{q}_n^*(\alpha,x) - q(\alpha,x)/g(x)| = o_p\left(\left(nh^{1+2\nu}\right)^{-1/2}\right)$. An assumption that the conceptual trimming parameter ζ_1 is sufficiently large so as to make

$$\sqrt{nh^{1+2\nu}}n^{-\zeta_1} = o(1) \tag{30}$$

has also been invoked.

The asymptotic normality of $\sqrt{nh^{1+2\nu}}\left(\bar{\theta}_n(\alpha,x) - E\left[D_x^\nu K_h\left(X_1 - x\right)\cdot\left(q\left(\alpha,X_1\right)/g\left(X_1\right)\right)\cdot\tilde{\tau}_n\left(g\left(X_1\right)\right)\right]\right)$ is then proved for fixed $(\alpha,x)\in\mathcal{A}\times\mathcal{X}$. In order to show this, it is sufficient that

$$\begin{aligned}
& h^{1+2\nu}Var\left[D_x^\nu K_h\left(X_1 - x\right)\frac{q\left(\alpha,X_1\right)}{g\left(X_1\right)}\cdot\tilde{\tau}_n\left(g\left(X_1\right)\right)\right] \\
\to\ & \Omega(\alpha,x)
\end{aligned}$$

for some $\Omega(\alpha,x)\in(0,\infty)$ along with

$$\begin{aligned}
& \frac{h^{2+4\nu}}{n}E\left[\left(D_x^\nu K_h\left(X_1 - x\right)\right)^4\frac{q^4\left(\alpha,X_1\right)}{g^4\left(X_1\right)}\cdot\tilde{\tau}_n^4\left(g\left(X_1\right)\right)\right] \\
=\ & o(1).
\end{aligned}$$

Note from (26) that

$$\begin{aligned}
& E\left[\left(D_x^\nu K_h\left(X_1 - x\right)\right)^2\frac{q^2\left(\alpha,X_1\right)}{g^2\left(X_1\right)}\tilde{\tau}_n^2\left(g\left(X_1\right)\right)\right] \\
=\ & h^{-1-2\nu}\left[\frac{q^2(\alpha,x)}{g(x)}\int\left(\frac{d^\nu}{dt_1^\nu}K\left(t_1\right)\right)^2 dt_1 + O\left(n^{-\zeta_1}+h\right)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& h^{1+2\nu} E\left[\left(D_x^\nu K_h\left(X_1-x\right)\right)^2 \frac{q^2\left(\alpha, X_1\right)}{g^2\left(X_1\right)} \tilde{\tau}_n^2\left(g\left(X_1\right)\right)\right] \\
= & \frac{q^2(\alpha, x)}{g(x)} \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^2 d t_1+O\left(n^{-\zeta_1}+h\right),
\end{aligned}$$

where the conditions that $\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}}\left|q^{(j)}(\alpha, x)\right|<\infty$ for $j \in\{0,1\}$; $\sup_{x \in \mathcal{X}}\left|g^{(j)}(x)\right|<\infty$ for $j \in\{0,1\}$; $\left|\int t_1\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^2 d t_1\right|<\infty$ and $\int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^2 d t_1<\infty$ have all been invoked.

As such, if in addition we invoke the condition that $\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}}\left|D_x^\nu q(\alpha, x)\right|<\infty$, we have

$$\begin{aligned}
& h^{1+2\nu} Var\left[D_x^\nu K_h\left(X_1-x\right) \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} \tilde{\tau}_n\left(g\left(X_1\right)\right)\right] \\
= & h^{1+2\nu}\left[E\left[\left(D_x^\nu K_h\left(X_1-x\right)\right)^2 \frac{q^2\left(\alpha, X_1\right)}{g^2\left(X_1\right)} \tilde{\tau}_n^2\left(g\left(X_1\right)\right)\right]\right. \\
& \left.-\left(E\left[D_x^\nu K_h\left(X_1-x\right) \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} \tilde{\tau}_n\left(g\left(X_1\right)\right)\right]\right)^2\right] \\
= & \frac{q^2(\alpha, x)}{g(x)} \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^2 d t_1+O\left(n^{-\zeta_1}+h\right) \\
& -h^{1+2\nu}\left[\left(D_x^\nu q(\alpha, x)\right)^2-D_x^\nu q(\alpha, x) \cdot D_x^\nu q^{(2)}(\alpha, x) \cdot h^2 \int t_1^2 K\left(t_1\right) d t_1\right. \\
& \left.+O\left(h^4+n^{-\zeta_1}\right)\right] \\
= & \frac{q^2(\alpha, x)}{g(x)} \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^2 d t_1+o(1) \\
\equiv & \Omega(\alpha, x)+o(1).
\end{aligned}$$

Next, note that

$$\begin{aligned}
& \frac{h^{2+4\nu}}{n} E\left[\left(D_x^\nu K_h\left(X_1-x\right)\right)^4 \frac{q^4\left(\alpha, X_1\right)}{g^4\left(X_1\right)} \tilde{\tau}_n^4\left(g\left(X_1\right)\right)\right] \\
= & \frac{h^{2+4\nu}}{n} \int\left(D_x^\nu K_h\left(x_1-x\right)\right)^4 \frac{q^4\left(\alpha, x_1\right)}{g^3\left(x_1\right)} \tilde{\tau}_n^4\left(g\left(x_1\right)\right) d x_1 \\
= & n^{-1} h^{-2} \int\left(\frac{d^\nu}{d h^{-1}\left(x_1-x\right)} K\left(h^{-1}\left(x_1-x\right)\right)\right)^4 \frac{q^4\left(\alpha, x_1\right)}{g^3\left(x_1\right)} \tilde{\tau}_n^4\left(g\left(x_1\right)\right) d x_1 \\
= & n^{-1} h^{-1} \int\left(\frac{d^\nu}{d t_1} K\left(t_1\right)\right)^4 \frac{q^4\left(\alpha, x+H t_1\right)}{g\left(x+H t_1\right)} \tilde{\tau}_n^4\left(g\left(x+H t_1\right)\right) d t_1 \\
= & n^{-1} h^{-1}\left[\frac{q^4(\alpha, x)}{g^3(x)} \tilde{\tau}_n^4(g(x)) \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^4 d t_1+O(H)\right] \\
= & n^{-1} h^{-1}\left[\frac{q^4(\alpha, x)}{g^3(x)} \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^4 d t_1+\frac{q^4(\alpha, x)}{g^3(x)}\left(\tilde{\tau}_n^4(g(x))-1\right) \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^4 d t_1\right. \\
& +O(h)] \\
= & n^{-1} h^{-1}\left[\frac{q^4(\alpha, x)}{g^3(x)} \int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^4 d t_1+O\left(n^{-\zeta_1}+h\right)\right] \\
= & o(1),
\end{aligned}$$

where the additional conditions that $\left|\int t_1\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^4 d t_1\right|<\infty$ and $\int\left(\frac{d^\nu}{d t_1^\nu} K\left(t_1\right)\right)^4 d t_1<\infty$ have been invoked.

Conclude via Liapounov's theorem that

$$\begin{aligned} & \sqrt{nh^{1+2\nu}} \left(\bar{\theta}_n(\alpha, x) - E \left[D_x^\nu K_h \left(X_1 - x \right) \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} \tilde{\tau}_n \left(g\left(X_1\right) \right) \right] \right) \\ \stackrel{d}{\to} \quad & N\left(0, \Omega(\alpha, x)\right). \end{aligned} \tag{31}$$

Finally, given the assumptions that $\sqrt{nh^{1+2\nu}} \cdot h = o(1)$, $\sup_{\alpha \in \mathcal{A}} \sup_{x \in \mathcal{X}} \left| D_x^\nu q^{(2)}(\alpha, x) \right| < \infty$ and the condition (30) on the conceptual trimming parameter ζ_1, we have the convergence

$$\begin{aligned} & \sqrt{nh^{1+2\nu}} \left(E \left[D_x^\nu K_h \left(X_1 - x \right) \frac{q\left(\alpha, X_1\right)}{g\left(X_1\right)} \tilde{\tau}_n \left(g\left(X_1\right) \right) \right] - D_x^\nu q(\alpha, x) \right) \\ = \quad & O\left(\sqrt{nh^{1+2\nu}} \left(h^2 + n^{-\zeta_1} \right) \right) \\ = \quad & o(1), \end{aligned} \tag{32}$$

from which it is deduced that the bias of $\bar{\theta}_n(\alpha, x)$ vanishes asymptotically.

Combine (29), (31) and (32) and deduce that $\sqrt{nh^{1+2\nu}} \left(\hat{\theta}_n(\alpha, x) - D_x^\nu q(\alpha, x) \right) \stackrel{d}{\to} N\left(0, \Omega(\alpha, x)\right)$ for each $(\alpha, x) \in \mathcal{A} \times \mathcal{X}$.

The second part of Theorem 3.3 follows from (24) and the conditions imposed on $\tilde{\tau}_n(\cdot)$ and ζ_1.

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