

Homework 2

Math 198: Math for Machine Learning

Due Date: February 19

Name:

Student ID:

Instructions for Submission

Please include your name and student ID at the top of your homework submission. You may submit handwritten solutions or typed ones (L^AT_EX preferred). If you at any point write code to help you solve a problem, please include your code at the end of the homework assignment, and mark which code goes with which problem. Homework is due by start of lecture on the due date; it may be submitted in-person at lecture or by emailing a PDF to both facilitators.

1 Comparing Vector Spaces

1. Exhibit a basis for $\mathbb{R}^3 := \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$.
2. Exhibit a basis for $\mathbb{P}^2 := \{f : \mathbb{R} \rightarrow \mathbb{R} : f(x) = a_0 + a_1x + a_2x^2 \text{ for some } a_0, a_1, a_2 \in \mathbb{R}\}$, the space of 3rd degree polynomials with real coefficients. Note that your basis elements should be polynomials.
3. Conclude that \mathbb{R}^3 and \mathbb{P}^2 are *isomorphic* (i) by a dimension argument and (ii) by exhibiting an *isomorphism* between them. When two vector spaces V, W are isomorphic, we write $V \cong W$.

2 Characterizing the Inner Product

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathbb{R}^n . Show that there exists $A \in \mathbb{R}^{n \times n}$ with $A^\top = A$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top A \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. (Hint: what is the action of $\langle \cdot, \cdot \rangle$ on the standard basis?)

3 Linear Maps

Let V, W be vector spaces, and let $T : V \rightarrow W$ be a linear map.

1. Show that T is one-to-one (a.k.a. injective) if and only if the kernel of T is trivial, i.e. $\{v \in V : T(v) = \mathbf{0}_W\} = \{\mathbf{0}_V\}$.
2. Let $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V , and let T be such that $\{T(\mathbf{b}_1), \dots, T(\mathbf{b}_n)\}$ is a basis for W . Show that T is an isomorphism.

4 Dual Spaces

Given a vector space V , we can form the vector space of linear maps from V to \mathbb{R} , called the *dual space* of V . Formally, the dual space is given by $V^* := L(V, \mathbb{R})$, and its elements are known as *linear functionals* (or, in some contexts, *covectors*).

1. Let V be a real vector space with basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Exhibit a basis for V^* and conclude that $V \cong V^*$.
2. (OPTIONAL) Consider V^{**} , the dual space of the dual space of V , called the *double dual space* of V . Without choosing a basis for V , construct an isomorphism between V and V^{**} . Since such an isomorphism exists, we say that V and V^{**} are *canonically isomorphic*.
3. (OPTIONAL) Let H be a (real) Hilbert space, i.e. an inner product space (perhaps infinite-dimensional) that is complete with respect to the metric induced by its inner product. Form the *continuous dual space* of H , $H' = \{\xi \in H^* : \xi \text{ is continuous}\}$. It turns out that H is canonically isomorphic to its continuous dual; this result in functional analysis known as the Riesz Representation Theorem. Give a guess as to the canonical isomorphism $H \rightarrow H'$. (Hint: It depends completely upon the coordinate-wise linearity of the inner product.)

5 Projections

A set of vectors is *orthogonal* if each vector is pairwise orthogonal to all the rest. An orthogonal set of vectors is *orthonormal* if each vector has norm 1 (i.e., $\langle \mathbf{v}, \mathbf{v} \rangle = 1$).

1. Show that any set of orthonormal vectors is linearly independent.
2. Show that, for a space V with dimension d , a set of d orthonormal vectors in V is a basis for V .
3. Suppose we have a vector space V , a subspace $W \subset V$, and a vector $\mathbf{v} \in V$ such that $\mathbf{v} \notin W$. Let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be an orthonormal basis for W .

(a) Show that

$$\mathbf{v}_w = \sum_{i=1}^k \langle \mathbf{v}, \mathbf{w}_i \rangle \mathbf{w}_i$$

is an orthogonal projection of \mathbf{v} into W .

- (b) Let \mathbf{P}_W be a matrix such that $\mathbf{P}_W \mathbf{v} = \mathbf{v}_w$. \mathbf{P}_W is known as a *projection matrix*.
- i. Show that $\mathbf{P}_W^2 = \mathbf{P}_W$.
 - ii. Show that $\mathbf{P}_W^\top = \mathbf{P}_W$.
 - iii. Show that $\mathbf{P}_W \mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in W$. Conclude that \mathbf{P}_W has the same action as the identity matrix \mathbf{I} on vectors in W .
4. Let $V = \mathbb{R}^4$ and $U \cong \mathbb{R}^2$ such that $\{e_1, e_2\}$ is a basis for U (i.e. all vectors in U take the form $(a, b, 0, 0)$ with respect to the standard basis for \mathbb{R}^4). Note that U is a subspace of V . Determine the matrix form of \mathbf{P}_U , and show that it has the same properties as \mathbf{P}_W from problem 3.